aspects of
QED renormalization
in anti-de Sitter space

aspects of **QED renormalization** *in anti-de Sitter space*

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preface

This thesis is about certain aspects of renormalization in anti-de Sitter spacetime, one of the simplest curved spaces imaginable. It focuses on quantum electrodynamics (QED), the theory of light and its interaction with matter. Unfortunately, it does not present a complete renormalization programme despite having a few giant shoulders to stand on. This, then, is another thesis among many, a work of more perspiration than inspiration about a rather obscure topic in a distant corner of physics. But the science of physics surely counts among the most impressive human achievements; if even one reader will take something useful from it, this thesis will be proud to be part of this, no matter how minor.

It may be a cliché, but none the less valid for that: like any other thesis, this one would not have been possible without the support and help of a long list of other people. My thanks and gratitude go to God for being the rock upon which I can safely build; to my colleagues for fruitful discussions; to my wife Inger for her support and the many personal sacrifices she made, the proofreading of an incomprehensible manuscript only being the least of them; to Willy and Peter, my mother and father; to Paule, my friends, and so many others — I cannot possibly mention all of you, but you know who you are.

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chapter 1

Introduction

Quantum field theory in curved space has long been an active field of research, studied for a variety of reasons. Some have regarded the field as a way to better understand physics in flat Minkowski space-time, by studying the limit of zero curvature or examining the additional constraints curvature imposes. Many, more ambitiously, have seen it as a step to a full theory of quantum gravity, or indeed used it as a vital ingredient of such a theory. For although the weak, electromagnetic and strong forces have been unified in the Standard Model – with unparalleled success, possible revision in the light of recently observed neutrino oscillations notwithstanding – it has so far proven impossible to incorporate the fourth fundamental force, gravitation, without ending up with a non-renormalizable and therefore meaningless theory. Attempts to unify these four forces, be it gauged supergravity, supersymmetric extensions to the Standard Model, or string theory, tend to be beset with incurable infinities, formal problems, as yet undiscovered particles, or apparently wrong physics.

One way to simplify the problem enough so that physically meaningful results might be extracted perturbatively is to embed the Standard Model in a static gravitational background, ignoring the dynamics of the gravitational field itself. This is not dissimilar to the familiar problem in quantum electrodynamics of a system in an electric or magnetic background field, where the fine structure constant α plays no part because the electromagnetic field produced by the system itself is neglected. In a gravitational background, then, G no longer plays a fundamental role. This is a much more tractable field of study and many areas have been mapped out well enough to be textbook material [BD82, Ful89].

Quantum field theory in a curved background is thought to provide an accurate description of reality above the Planck scale. Re-evaluating some of the core theoretical results in the presence of such a background would therefore be a fruitful way to gauge the effects of gravitation on elementary processes above this scale. However, many of the calculations that are standard fare in flat space prove impossible to do analytically in a general curved space. More simplification is needed.

The simplest possible generalization of Minkowski space-time that still has a lower bound on the energy is anti-de Sitter space. It is a maximally symmetric space, where curvature is constant and all points in the space-time manifold are completely equivalent. Although gravitational backgrounds in nature are generally more complicated than this, such a simplified anti-de Sitter physics may help develop the research tools and insights to deal with more general cases later. Not that it is of purely theoretical interest only. Physics in a maximally symmetric gravitational background field may be a useful first approximation for the way in which the energy-momenta of the particles locally curve space in high-energy scattering processes. The particular space considered here – anti-de Sitter space – has also found applications in phenomenological bag models, ranging from the Nijmegen hadron bag [vBDR84] modeling quark confinement to an extension of the Skyrme nucleon model [Ros93], and plays a role in many of the fundamental theories proposed today.

The present thesis makes a start with the renormalization of quantum electrodynamics (QED) in an anti-de Sitter background space-time. It is therefore not an attempt to directly formulate a fundamental theory of matter, but a study of the effects of gravitation on some well-understood physics above the Planck scale. In a sense, it fits in, if only perhaps as a footnote, with the efforts initiated by Fronsdal et al in the sixties [Fro65], which may be said to have their roots in Dirac's own work [Dir36] thirty years before. The ultimate aim of the programme followed is first, to come to a full treatment of quantum electrodynamics (QED) in anti-de Sitter space much as Drummond and Shore have done for spherical space-time [DS79] and second, to determine the effects of gravitation upon physical observables such as the anomalous magnetic momentum of the electron. Ultimately, the entire Standard Model could be formulated in this way. Unfortunately, the inevitable difficulties and time constraints meant that only the first cautious steps have been taken towards that goal.

An overview is given of d-dimensional anti-de Sitter space, its $SO(d-1,2)$ symmetry group, and its realization as an embedded manifold in $\mathbb{E}(d-1,2)$. The embedding procedure is briefly developed for more general manifolds, as many standard texts about physics in curved space do not discuss this in depth. After recalling the construction of the anti-de Sitter scalar propagator [DvB85, FD91], existing results for its massive spinor and vector propagators [JD86, JD87] are generalized to $4+\epsilon$ dimensions, with some remarks about the massless limit. It will be shown how they assume the familiar Hadamard form in the $d \to 4$ limit only through mutually canceling infinities. It becomes explicit that the simplification of the anti-de Sitter vector propagator in the $c = 1/3$ gauge does not appear for $d \neq 4$ because of the appearance of anomalous mass terms due to the loss of conformal invariance. Using the dimensional regularization scheme [tHV72] in configuration space, the infinities of the three fundamental QED diagrams: vacuum polarization, electron self-energy and the vertex diagram, are calculated; to aid with the latter, its configuration space Ward-Takahashi identity is derived. It will be shown that up to first order the theory is renormalizable and gives rise to the rescaling factors familiar from flat space QED. It will be discussed how the renormalization of quantum field theory in anti-de Sitter space can be further developed and how results for the finite parts might be obtained.

The next chapter discusses anti-de Sitter space, the representations of its symmetry group, and its embedding. In chapter 3, propagator expressions will be derived. Some attention will be paid to the massless limit. With the basic groundwork laid, chapter 4 will focus on the three fundamental one-loop QED processes. Finally, conclusions and perspectives for future research will be discussed.

chapter 2

Anti-de Sitter space

De Sitter space, and its negative curvature counterpart anti-de Sitter space, are the simplest curved spaces one can study thanks to their high degree of – in fact maximal – symmetry. In the de Sitter universe, every point is created equal. As such, they were the first to be studied, being introduced into the field of physics by de Sitter as early as 1917 [dS17b, dS17a] and belong to the staple diet of textbooks [Wei72, MTW73, HE73, BD82, to name just a few]. For spaces so old and well-studied, they have proven to be remarkably resilient, coming back in vogue every time physical fashion threatened to abandon them to irrelevance.

The uses of (anti-)de Sitter spaces, then, are many. They are solutions of the Einstein equations for empty space in the presence of a cosmological constant. Observations teach that this constant must be extremely small, but if it is nonzero there is a universal background curvature. Dirac proposed in 1935 that field theories should be invariant under the de Sitter groups instead of the Poincaré group because of the expansion of the universe [Dir36]. Inflationary cosmological models indicated the universe may have gone through a de Sitter era. Anti-de Sitter space turned out to be more physically viable than de Sitter space because of the latter's indefinite Hamiltonian spectrum. The space has been regarded as a way to gain insight into ordinary quantum field theory because of its well-defined Inönü-Wigner contraction to flat space [IW53], notably by Fronsdal et al [Fro65, Fro74, FH75, BFH83]. It has also been hoped that its natural length scale can be used to regularize infrared divergences. De Sitter spaces have arisen in studies of gauged quantum gravity – in a sense, the (anti-) de Sitter group arises as the minimal extension of the Poincaré gauge group necessary to make it renormalizable – and higher-spin fields, and play a role in some string theories [Duf98, and references therein]. The distortion of space-time by high energy particles can possibly be approximated locally by a symmetric, constant curvature background.

Even if anti-de Sitter space would play no part in fundamental theory, it is a highly symmetrical space where a wide range of computations is still tractable. Moreover, the movement of a free particle in anti-de Sitter space resembles that of a harmonic oscillator [DvB83]. These properties have been put to good use in phenomenological models such as bags – the Nijmegen quark bag [vBDR84] among them – and as an extension to Skyrme models [Ros93, Ros94]. The latter and many other adS bag models are examples of "Cheshire cat" bags, where the physics is ultimately independent of the bag parameters: the "unphysical" bag effectively vanishes, leaving only its "grin" behind in the form of computational advantages. Large-N gauge theory may even effectively lead to an adS Cheshire cat bag ([Vol98] and references therein), where the bag is a consequence of a fundamental theory rather than a purely artificial construct.

This section pulls together various aspects and properties of anti-de Sitter space that are used in the remainder of this thesis, such as its definition, representations, embedding, the covering space and the two-point invariant λ.

2.1 A first glance

Take a maximally symmetric, d-dimensional, curved space $\mathcal{N}$ with Minkowski signature $(-1, -1, \ldots, -1, +1)$. The metric is uniquely determined [Wei72, HE73, BD82]: in terms of a coordinate system $x = (\Omega, r, t)$, where Ω is a collection of angular variables, $r^2 = \vec{x}^2$, $t = x^d$,

$$ds^2 = (1 + \alpha r^2)dt^2 - (1 + \alpha r^2)^{-1}dr^2 - r^2 d\Omega^2 \tag{2.1}$$

or, in terms of a simple "Cartesian" coordinate system x^μ,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.2}$$

$$[g_{\mu\nu}] = \begin{pmatrix} -\delta_{kl} + \frac{\alpha}{1+\alpha r^2}x^k x^l & 0 \\ 0 & 1 + \alpha r^2 \end{pmatrix} \tag{2.3}$$

These spaces are solutions of the Einstein equations for empty space with a cosmological constant $\Lambda = -(d-1)(d-2)\alpha/2$:

$$\mathcal{E}_{\mu\nu} \stackrel{\text{def}}{=} \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -\Lambda g_{\mu\nu}. \tag{2.4}$$

They can be classified according to the value of this constant:

- $\Lambda > 0$: de Sitter space,
- $\Lambda = 0$: Minkowski space, and
- $\Lambda < 0$: anti-de Sitter space.

As can be expected from maximally symmetric spaces, they are homogeneous and isotropic about all points. Their curvature is constant:

$$\begin{aligned} \mathcal{R}_{\mu\nu\rho\sigma} &= \frac{2\Lambda}{(d-1)(d-2)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &= -\alpha(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ \mathcal{R}_{\mu\nu} &= \frac{2\Lambda}{d-2}g_{\mu\nu} = -(d-1)\alpha g_{\mu\nu} \\ \mathcal{R} &= \frac{2d\Lambda}{d-2} = -d(d-1)\alpha \end{aligned} \tag{2.5}$$

These curvature tensors, of course, work out to be zero in the $\Lambda = 0$ Minkowski case. The expression (2.5) for $R_{\mu\nu\rho\sigma}$ shows that the Weyl tensor $C_{\mu\nu\rho\sigma}$ is zero: these spaces are conformally flat.

2.2 The anti-de Sitter group

In anti-de Sitter space $\mathcal{N}$, the kinematical Poincaré group $\mathrm{P}_+^\uparrow(d-1,1)$ of Minkowski space is replaced by $\mathrm{SO}_0(d-1,2)$ — this will become readily apparent when the space is realized as an embedded hyperboloid in section 2.4. The space can be represented as the coset space $\mathrm{SO}(d-1,2)/\mathrm{SO}(d-1,1)$. The Lie algebra $\mathrm{so}(d-1,2)$ is generated by M_{MN}:

$$[M_{KL}, M_{MN}] = -i(\eta_{KM}M_{LN} + \eta_{LN}M_{KM} - \eta_{KN}M_{LM} - \eta_{LM}M_{KN}) \tag{2.6}$$

where $\eta = \mathrm{diag}(-1,\dots,1,+1,+1)$ is $d+1$ dimensional.

There is a well known continuous limit from (anti-)de Sitter space to Minkowski space, the Inönü-Wigner contraction [IW53]: for $\Lambda \to 0$, the curvature $\mathcal{R}$ becomes zero and $g_{\mu\nu}$ becomes the Minkowski metric $\eta_{\mu\nu}$. Given the parallel between anti-de Sitter space $\mathcal{N} = \mathrm{SO}(d-1,2)/\mathrm{SO}(d-1,1)$ and Minkowski space $M_d = \mathrm{P}_+^\uparrow(d-1,1)/\mathrm{SO}(d-1,1)$ it is clear that – labelling the highest ($d+1$th) index 5 for the sake of convenience – the generators $M_{5\mu}$ can in the contraction be reinterpreted as the translation generators P_μ: redefining

$$\sqrt{\alpha}M_{5\mu} \stackrel{\text{def}}{=} P_\mu, \text{ where } \mu = 1\dots d \tag{2.7}$$

the algebra (2.6) becomes

$$\begin{aligned}
[P_\mu, P_\nu] &= -i\alpha M_{\mu\nu} \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma}) \\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\nu)
\end{aligned} \tag{2.8}$$

It is evident that for $\Lambda, \alpha \to 0$ the translation generators P_μ become a commutative subalgebra, the remaining $\mathrm{so}(d-1,1)$ subalgebra is preserved in the contraction and the familiar Poincaré algebra is recovered. The generators $M_{\mu\nu}$, where $\mu,\nu = 1\dots d$, generate Lorentz transformations.

The representations of $\mathrm{SO}(3,2)$ have been studied for decades [Fro65, Eva67]; the study of $\mathrm{SO}(d-1,2)$ representations for arbitrary d continues today. The Casimir operators of the group are

$$\begin{aligned}
C_1 &= \tfrac{1}{2}M_{MN}M^{MN} \\
C_2 &= W^2, \text{ where } W_J = \tfrac{1}{8}\epsilon_{JKLMN}M^{KL}M^{MN}
\end{aligned} \tag{2.9}$$

In the contraction, these become the familiar $C_1 \sim P^2$ and $C_2 \sim W^2$ where W^μ is the Pauli-Lubanski vector. In $d = 4$, the group allows four classes of unitary irreducible representations:

- The identity representation;

- With a lower bound $m > 0$ for P_4 (the "energy"), M_{12} taking eigenvalues $-j \dots j$;
- As before but with an upper bound $-m$ for P_4;
- Representations with unbounded energy and spin.

In the context of this thesis, the most interesting representations are those with a positive energy m. The structure can be clarified further by rewriting the generators and considering their $\Lambda \to 0$ contraction. The generator $E = M_{54}$ is proportional to P_4 and represents the energy; for $\Lambda < 0$ it is a compact group and its eigenvalues are discrete. Still for $d = 4$, the subalgebra $J_k = \frac{1}{2}\epsilon_{klm}M_{lm}, k = 1 \dots 3$, satisfies the commutation relations of so(3) and can be interpreted as the generators of spatial rotations. In general, $\mathrm{SO}(d-1,2)$ is locally isomorphic to $\mathrm{Sp}(d)/\mathbb{Z}_2$ and allows spin representations.

The eigenvalues of the quadratic and quartic Casimir operators C_1 and C_2 are

$$\begin{aligned} \alpha C_1 &= E_0(E_0 - 3) + m_1(m_1 + 1) \\ \alpha C_2 &= -(E_0 - 1)(E_0 - 2)m_1(m_1 + 1) \end{aligned} \tag{2.10}$$

For arbitrary d, C_1 extends to [Met95, Met98, and references contained therein]:

$$\alpha C_1 = E_0(E_0 - d + 1) + \sum_{l=1}^{\nu} m_l(m_l - 2l + d - 1)$$

with a total of $\nu = (d-2)/2$ parameters m_l. Irreducible representations are specified by the minimum energy and the m_l: $D(E_0, \{m_l\})$.

For $d = 4$, the parameters $\{m_l\}$ reduce to just one parameter, the spin s. One can then go on to define states in Hilbert and Fock spaces in the usual way [Fro74, FH75]; the construction of $\mathcal{H}_{E_0,s}$ in terms of coherent fields [AAGM95, SS99] is especially interesting and may enable further development of the material presented in this thesis. Writing $E_0(E_0 + d - 1) = \alpha^{-1}m_0^2$, we get in the contraction to Minkowski space:

$$\begin{aligned} C_1 &= m_0^2 \\ C_2 &= -s(s+1)m_0^2 \end{aligned} \tag{2.11}$$

confirming the interpretation of E_0 and s.

It is well known that for $\mathrm{SO}(3,2)$ fields, the representations are unitary iff $E_0 \geq s + 1$ [Eva67], with the exception of the two Dirac singleton representations $D(\frac{1}{2}, 0)$ and $D(1, \frac{1}{2})$ [Dir63, FF81], which have no Minkowski space analogues. If equality holds, these $D(s+1, s)$ fields are massless [Fro79, FF80a].

For arbitrary d, however, there are many massless representations, among which are a totally symmetric representation $D(s+d-3, s, 0, \ldots, 0)$ and, iff $s \leq \nu$, a totally antisymmetric one which is $D(d-1-s, 1, \ldots, 1, 0, \ldots, 0)$ for bosonic fields and $D(d-\frac{1}{2}-s, \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ for fermionic ones, where the 1 and $\frac{3}{2}$ occur s times. The unitarity condition becomes more complicated as well, where the minimum value of E_0 depends on $\{m_l\}$; see Metsaev [Met95, Met98] for a discussion. For $d \geq 6$ these massless fields have a nontrivial flat space limit [Met00]. In the context of the renormalization of $SO(3,2)$ QED, however, the primary interest is in the $d \sim 4$ region.

Both in order to get a clear conceptual view of anti-de Sitter space and to perform practical computations, it will prove fruitful to realize the space by embedding it as a curved surface in a higher dimensional, pseudo-Euclidean space. Before going into the specifics, a brief excursion will be made into the general procedure of embedding.

2.3 Embedding maps

Consider a d-dimensional smooth Riemannian metric manifold $\mathcal{N}$[1]. It is a well-known fact that it can generally be embedded in a higher-dimensional (pseudo-)Euclidean manifold. The lowest possible dimension for this embedding manifold depends on the differentiable character of the embedding functions; establishing this dimension is an important topic in differential topology but unfortunately outside the scope of this thesis. Two important results are that if the manifold has dimension $d \geq 3$ and the embedding functions are differentiable, then $d' \leq d(d+3)/2$; if they are analytic, then $d' \geq d(d+1)/2$. Call the embedding manifold $\mathcal{M}$, with d' dimensions, n' of which have a negative signature: its structure is $\mathbb{E}(n', d'-n')$. In $\mathcal{M}$, erect coordinates y^M such that the metric is diagonal: $\eta_{MN} = \mathrm{diag}(-1, \dots, -1, +1, \dots, +1)$, with n' times -1 and $d'-n'$ times $+1$. The embedding can now be parametrized as follows:

$$\mathcal{N} = \{y \in \mathcal{M} | \forall_{k=1\dots(d'-d)} : f^k(y) = 0\} \tag{2.12}$$

For each $y \in \mathcal{M}$, there is a tangent space $T(y, \mathcal{M})$ spanned by $\{\partial_M\}$ and a cotangent space $T^*(y, \mathcal{M})$ spanned by $\{\mathrm{d}y^M\}$. Restricting attention to the submanifold $\mathcal{N}$, it is possible for each $y \in \mathcal{N}$ to find an $\mathrm{SO}(n', d'-n')$ transformation $G(y)$ such that $T^*(y, \mathcal{N})$ is spanned by the first d basis covectors of the transformed $T^*(y, \mathcal{M})$:

$$\forall_{y\in\mathcal{N}} \exists_{G\in\mathrm{SO}(n,d'-n)} : T^*(y, \mathcal{N}) = \{G^a{}_N \mathrm{d}y^N\},\ a = 1 \dots d. \tag{2.13}$$

This transformation is unique up to an $\mathrm{SO}(n, d-n)$ transformation leaving $T^*(y, \mathcal{N})$ invariant. A similar expression follows for the tangent space:

$$T(y, \mathcal{N}) = \{G^{-1N}{}_a(y)\partial_N\}, \tag{2.14}$$

where $G^{-1N}{}_A = G_A{}^N$ since $G \in \mathrm{SO}(n, d'-n)$. Finally, it is worth noting that the 1-forms $\mathrm{d}f^k(y)$ generate the cotangent space $T^*(y, \mathcal{M}/\mathcal{N})$ complementing $T^*(y, \mathcal{N})$, so that

$$\forall_{y\in\mathcal{N}} : T^*(y, \mathcal{M}) = T^*(y, \mathcal{N}) \otimes \{\mathrm{d}f^k(y)\};$$

indeed, one can identify

$$\{\mathrm{d}f^k(y)\} = \{G^{\bar{a}}{}_N \mathrm{d}y^N\},\ \ \bar{a} = (d+1)\dots d'.$$

[1]For a thorough review of the techniques and notations used in differential geometry and topology in physics, see Eguchi, Gilkey and Hanson [EGH80]. This section and the thesis in general necessarily gloss over the bulk of the detail and structure to be found here.

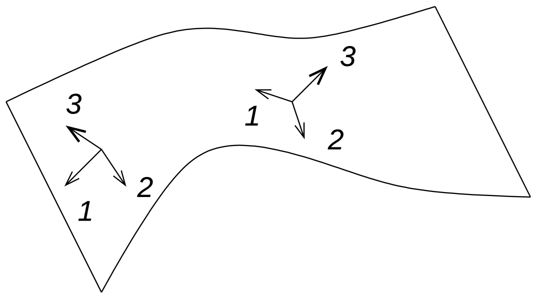


figure 2.1: $G^N{}_A\partial_N$ in three dimensions; the first two legs span $\mathcal{N}$.

In other words, the cotangent space spanned by the exterior derivative of the $d - d'$ functions $f^k(y)$ is equal to the space complementary to $T^*(y, \mathcal{N})$. This follows trivially from the parameterization (2.12) of $\mathcal{N}$.

Now coordinates x^μ, $\mu = 1 \dots d$, on the embedded manifold $\mathcal{N}$, will be introduced where y can be written as a smooth function of these coordinates: $y = y(x)$. The metric of $\mathcal{N}$ in this system can then be expressed

$$\begin{aligned} g_{\mu\nu}(x) &= \partial_\mu y^M(x)\partial_\nu y^N(x)\eta_{MN} = y^M{}_{,\mu} y^N{}_{,\nu}\eta_{MN} \\ \forall_{x\in\mathcal{N}} &\exists g^{\mu\nu} : g^{\mu\nu} g_{\nu\rho}(x) = \delta^\mu_\rho \end{aligned} \tag{2.15}$$

where a shorthand $\partial_\mu y(x)^M = y^M{}_{,\mu}$ has been introduced. This form of the metric is the consequence of the natural push-up map

$$\begin{aligned} y^M{}_{,\mu}(x) : T(x, \mathcal{N}) &\to T(y(x), \mathcal{M}) \\ v^\mu(x) &\to v^M(y(x)) = y^M{}_{,\mu}(x) v^\mu(x) \end{aligned} \tag{2.16}$$

turning contravariant vectors in $\mathcal{N}$ to vectors in $\mathcal{M}$. Similarly, there is a pull-down map

$$\begin{aligned} y^M{}_{,\mu}(x) : T^*(y(x), \mathcal{M}) &\to T^*(x, \mathcal{N}) \\ w_M(y(x)) &\to w_\mu(x) = y^M{}_{,\mu}(x) w_M(y(x)) \end{aligned} \tag{2.17}$$

transporting covariant quantities from $\mathcal{M}$ to $\mathcal{N}$. Obviously, the tensor $y^M{}_{,\mu}$ has $d' \times d$ dimensions and therefore does not have an inverse. However, since this is a metric space its tensor indices can of course be lowered and raised respectively:

$$y_M{}^{,\mu} \stackrel{\text{def}}{=} g^{\mu\nu} y^N{}_{,\nu}(x)\eta_{NM}$$

which enables one to define two further maps

$$v^M(y(x)) \to v^\mu(x) = y_M{}^{,\mu}(x) v^M(y(x)) \tag{2.18}$$

$$w_\mu(x) \to w_M(y(x)) = y_M{}^{,\mu}(x) w_\mu(x) \tag{2.19}$$

and migrate arbitrary tensors freely between $\mathcal{N}$ and the hypersurface of $\mathcal{N}$ in $\mathcal{M}$, if not always without loss of information.

It is possible to define a projection operator $P : T^*(y, \mathcal{M}) \to T^*(y, \mathcal{N})$:

$$\begin{aligned} P^M{}_N &= G^{-1}{}^M{}_a G^a{}_N \\ &= y^M{}_{,\mu} g^{\mu\nu} y^K{}_{,\nu} \eta_{KN}. \end{aligned} \tag{2.20}$$

The equality of these two expressions can be readily proven by comparing their kernel $T^*(y, \mathcal{M}/\mathcal{N})$ and their dimensionality. Apart from the usual relations for projection operators, an important property of P in subsequent computations will be that it leaves $G^a{}_N$ invariant

$$G^a{}_K P^K{}_N = G^a{}_N, \ a = 1 \ldots d.$$

which is a consequence of Eqn. (2.13). Of course P leaves $y^M{}_{,\mu}$ invariant as well.

From equations such as (2.13) and (2.20) and the figure 2.3 it is clear that G appears to be a generalization of the vielbein for $\mathcal{N}$ extended to $\mathcal{M}$. Indeed, the vielbein and inverse vielbein of $\mathcal{N}$ are

$$\begin{aligned} e^a{}_\mu &= G^a{}_M y^M{}_{,\mu} \\ E^\mu{}_a &= g^{\mu\nu} e^b{}_\nu \eta_{ab} \end{aligned} \tag{2.21}$$

where $\eta_{ab} = \mathrm{diag}(-1, -1, \ldots, +1, +1)$, with n times -1 and $d - n$ times $+1$. This can be seen from

$$\begin{aligned} e^a{}_\mu e^b{}_\nu \eta_{ab} &= G^a{}_M G^b{}_N y^M{}_{,\mu} y^N{}_{,\nu} \eta_{ab} = P_{MN} y^M{}_{,\mu} y^N{}_{,\nu} \\ &= \eta_{MN} y^M{}_{,\mu} y^N{}_{,\nu} = \eta_{\mu\nu}. \end{aligned}$$

The Levi-Civita connection is

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma} \left[g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right] \\ &= y_M{}^{,\rho} y^M{}_{,\mu\nu}. \end{aligned} \tag{2.22}$$

From these, the spin connection can be readily computed:

$$\begin{aligned} \omega^a{}_{b\mu} &= e^a{}_\nu \mathcal{D}_\mu E^\nu{}_b = e^a{}_\nu \left[\delta^\nu_\rho \partial_\mu + \Gamma^\nu_{\rho\mu} \right] E^\rho{}_b \\ &= G^a{}_M y^M{}_{,\nu} \left[g^{\nu\sigma} y^K{}_{,\sigma} \eta_{KL} y^L{}_{,\rho} \partial_\mu \right. \\ &\qquad \left. + g^{\nu\sigma} y^K{}_{,\sigma} \eta_{KL} y^L{}_{,\rho\mu} \right] g^{\rho\delta} y^N{}_{,\delta} \eta_{NO} G^{-1}{}^O{}_b \\ &= G^a{}_M P^M{}_L \partial_\mu P^L{}_O G^{-1}{}^O{}_b \\ &= G^a{}_M \partial_\mu G^{-1}{}^M{}_b \end{aligned} \tag{2.23}$$

This completes the apparatus describing the embedded manifold $\mathcal{N}$.

The reason why it can be advantageous to embed $\mathcal{N}$ is that computations can simplify considerably if they are expressed in terms of objects in the embedding space $\mathcal{M}$. The embedding of tensors from $\mathcal{N}$ in $\mathcal{M}$ has already been discussed; tensor fields $v^{\mu\cdots}_{\nu\cdots}(x)$ in $\mathcal{N}$ can be extended to fields $v^{M\cdots}_{N\cdots}(y)$, $y \in \mathcal{M}$ by giving some prescription for their coordinate dependence outside the surface of $\mathcal{N}$.

Consider the covariant derivative of a vector field in $\mathcal{N}$

$$\mathcal{D}_\nu v^\mu \stackrel{\text{def}}{=} v^\mu_{;\nu} = v^\mu_{,\nu} + \Gamma^\mu_{\nu\rho} v^\rho.$$

Pushing this 2-tensor to $\mathcal{M}$ gives, using Eqn. (2.22),

$$\begin{aligned} y^M{}_{,\mu} y_N{}^{,\nu} v^\mu_{;\nu} &= y^M{}_{,\mu} y_N{}^{,\nu} v^\mu_{,\nu} + y^M{}_{,\mu} y_N{}^{,\nu} y_K{}^{,\mu} y^K{}_{,\nu\rho} v^\rho \\ &= y^M{}_{,\mu} \mathcal{D}_N v^\mu + P^M{}_K (\mathcal{D}_N y^K{}_{,\rho}) v^\rho \\ &= P^M{}_K \mathcal{D}_N v^K \end{aligned}$$

where $\mathcal{D}_M \stackrel{\text{def}}{=} y_M{}^{,\mu}\partial_\mu$, an operator in $\mathcal{M}$ often much simpler in form than the covariant derivative $\mathcal{D}_\nu$ in $\mathcal{N}$. During the construction of the spinor propagator in section 3.2 this turns out to be true for the spin-$\frac{1}{2}$ covariant derivative as well. In general it can be shown that for an (m, n) tensor

$$\begin{aligned} & y_N{}^{,\kappa} y^{M_1}{}_{,\mu_1} \cdots y^{M_m}{}_{,\mu_m} y_{N_1}{}^{,\nu_1} \cdots y_{N_n}{}^{,\nu_n} \mathcal{D}_\kappa T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \\ &= P^{M_1}{}_{M_1'} \cdots P^{M_m}{}_{M_m'} P^{N_1'}{}_{N_1} \cdots P^{N_n'}{}_{N_n} \mathcal{D}_K T^{M_1'\cdots M_m'}_{N_1'\cdots N_n'} \end{aligned} \tag{2.24}$$

so in general $\{\Pi P\}\mathcal{D}_M$ is the embedding in $\mathcal{M}$ of the covariant derivative $\mathcal{D}_\mu$ acting on tensor fields in $\mathcal{N}$. The commutator

$$[\mathcal{D}_M, \mathcal{D}_N] = y_{[M}{}^{,\nu} y_{N]}{}^{,\mu}{}_{,\nu} y^K{}_{,\mu} \partial_K$$

in particular is directly related to the curvature of $\mathcal{N}$.

2.4 Embedding anti-de Sitter space

Using the toolbox developed in the preceding section it is now possible to represent the spaces discussed in section 2.1 by embedding them in a pseudo-Euclidean space $\mathcal{M}$ of dimension $d' = d + 1$. Anti-de Sitter space, in particular, can be embedded in $\mathcal{M} = \mathbb{E}(d-1, 2)$ as the hyperboloid [Wei72]

$$f(y) = \eta_{MN} y^M y^N - R^2 = 0 \tag{2.25}$$

where $\eta = \mathrm{diag}(-1, \ldots, -1, +1, +1)$. This hyperboloid can be visualized, if two out of three dimensions are suppressed, as a two-dimensional hyperboloid embedded in three-dimensional space. Geodesic paths are traced by intersections of planes with this hyperboloid.

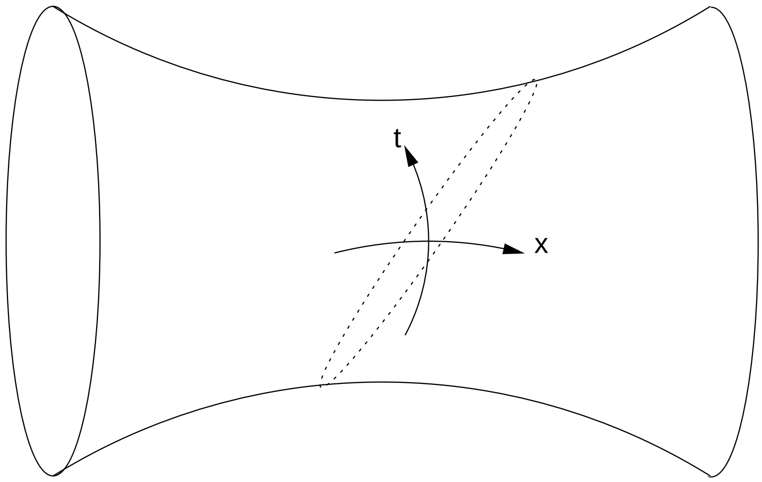


figure 2.2: Anti-de Sitter space. The dotted line traces a closed timelike geodesic.

The metric tensor (2.15) and curvature of this embedded manifold are easily computed:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{R^2 + r^2}{R^2}\mathrm{d}t^2 - \frac{R^2}{R^2 + r^2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{2.26}$$

$$\mathcal{R}(y) = \mathcal{R} = -d(d-1)R^{-2} \tag{2.27}$$

The space corresponds to (2.1) with $R^2 = \alpha^{-1} = -(d-1)(d-2)/2\Lambda$. An explicit expression for the transformation G defined as in (2.13) can be found:

$$[G^A{}_M(y)] = \begin{pmatrix} \delta^{kl} + \frac{y^k y^l}{R(R+\rho)} & -\frac{y^k y^4}{R\rho} & -\frac{y^k y^5}{R\rho} \\ 0 & \frac{y^5}{\rho} & -\frac{y^4}{\rho} \\ -\frac{y^l}{R} & \frac{y^4}{R} & \frac{y^5}{R} \end{pmatrix} \tag{2.28}$$

where $\vec{y}^2 = \Sigma_{k=1\ldots d-1}(y^k)^2, \rho^2 = R^2 + \vec{y}^2$ (the dimension d is arbitrary — the dth and $d+1$th component of y are denoted y^4 and y^5 as a notational convenience only). G is an $\mathrm{SO}(d-1,2)$ transformation such that $G^A_M(y)y^M = y^A_0 = (\vec{0}, 0, R)$. As mentioned in the previous section, there are many possible choices for G, differing only by an $\mathrm{SO}(d-1,1)$ Lorentz transformation.

The projection operator $P : T^*(y, \mathcal{M}) \to T^*(y, \mathcal{N})$ (2.20) is

$$\begin{aligned} P^{MN}(y) &= G^{-1M}{}_a(y)G^{aN}(y) \\ &= \eta^{MN} - \hat{y}^M\hat{y}^N \end{aligned} \tag{2.29}$$

where the second definition with

$$\hat{y}^M \stackrel{\text{def}}{=} \frac{y^M}{|y|}, \ |y| \stackrel{\text{def}}{=} \sqrt{y^2}.$$

extends the domain from $\mathcal{N}$ to $\mathcal{M}$ in an obvious way by imposing a zero degree of homogeneity. Expressions for the anti-de Sitter vielbein e^a_μ and the spin connection $\omega^a{}_{\mu\nu}$ follow readily using (2.21) and (2.23) from the preceding section. There is, in fact, a direct relation between the two:

$$e^a_\mu(y) = G^a{}_M(y)y^M{}_{,\mu} = R^{-1}\omega^a{}_{5\mu}(y). \tag{2.30}$$

The covariant derivative $\mathcal{D}_\mu$, when embedded in $\mathbb{E}(d-1,2)$ (see 2.24), simply becomes the tangential derivative $\mathcal{D}_M$

$$\mathcal{D}_M = P^N{}_M(y)\partial_N = y_M{}^{,\mu}\partial_\mu = \partial_M - \hat{y}_M\hat{y}\cdot\partial \tag{2.31}$$

an expression independent of the rank of the tensor it acts upon. The commutator of the $\mathcal{D}_M$s is proportional to the set of $d(d+1)/2$ Killing vectors:

$$[\mathcal{D}_M, \mathcal{D}_N] = R^{-1}\hat{y}_{[M}\partial_{N]} = -iR^{-2}M_{MN} = i\frac{\mathcal{R}}{d(d-1)}M_{MN} \tag{2.32}$$

where $M_{MN} \stackrel{\text{def}}{=} iy_{[M}\partial_{N]} \stackrel{\text{def}}{=} i(y_M\partial_N - y_N\partial_M)$ represents the generalized angular momentum operator in $\mathbb{E}(d-1,2)$. The M_{MN} form a representation of $\mathrm{so}(d-1,2)$ and can therefore, in the absence of spin part, be identified with the generators in (2.6). Between them, the operators $\mathcal{D}_M$ and M_{MN} form a representation of the conformal algebra so(d, 2) [Fro75]:

$$[\mathcal{D}_K, M_{MN}] = i\eta_{K[M}\mathcal{D}_{N]} \tag{2.33}$$

as is easily verified using Eqns. (2.32) and (2.6). The square of M can be expressed in terms of the square of the tangential derivative:

$$M^2 \stackrel{\text{def}}{=} -\frac{1}{2}M_{MN}M^{MN} = -y^2\mathcal{D}^2 \tag{2.34}$$

a relation which will be useful in practical computations (it is $\stackrel{\text{eff}}{=} -y^2\Box^2$ when working on expressions with zero degree of homogeneity, i.e. in the kernel of $T(y, \mathcal{M}/\mathcal{N})$).

In the embedding presented here, the Inönü-Wigner contraction corresponds to taking the $R \to \infty$ limit. In this limit, $P_{MN} \to \text{diag}(-1, \ldots, -1, +1, 0)$ essentially becomes the Minkowski metric, the extra embedding dimension losing all significance. It can now easily be verified that the reinterpretation of $M_{5\mu}$ as P in (2.7) does indeed give the familiar translation operator:

$$\lim_{R\to\infty} R^{-1} M_{5\mu} = i\partial_\mu \stackrel{\text{def}}{=} P_\mu \tag{2.35}$$

The two-point invariant and a coordinate map

Finally, the coordinate dependency of two-point functions invariant under $\mathrm{SO}(d-1, 2)$, i.e. invariant under M, must be expressible in terms of some two-point invariant quantity $\lambda(y_1, y_2)$ where $y_1, y_2 \in \mathcal{M}$. Starting with the invariant distance (2.2) in $\mathcal{N}$, embedding this in $\mathcal{M}$,

$$\begin{aligned} \mathrm{d}s^2(x) &= g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= P_{MN}(x)\mathrm{d}y^M(x)\mathrm{d}y^N(x) \end{aligned}$$

one can examine this quantity for finite $\mathrm{d}y \to \varepsilon \stackrel{\text{def}}{=} y_1 - y_2$ and see if it yields a useful generalization:

$$\begin{aligned} \lambda(y_1, y_2) &\stackrel{\text{def}}{=} P_{MN}(y_1)\hat{\varepsilon}^M\hat{\varepsilon}^N = P_{MN}(y_2)\hat{\varepsilon}^M\hat{\varepsilon}^N && (2.36) \\ &= 1 - z^2(y_1, y_2) \text{ where } z(y_1, y_2) \stackrel{\text{def}}{=} \hat{y}_1^M \hat{y}_{2M} && (2.37) \\ &= \hat{\varepsilon}^2 - \frac{1}{4}(\hat{\varepsilon}^2)^2 && (2.38) \end{aligned}$$

where $\hat{\varepsilon} \stackrel{\text{def}}{=} \hat{y}_1 - \hat{y}_2$ has been normalized to turn λ into a dimensionless quantity independent of R in both y_1 and y_2, i.e., $\forall_{O(y_1) \in T(y_1, \mathcal{M}/\mathcal{N})}$: $O(y_1)\lambda(y_1, y_2) = 0$. Since y^M can be readily expressed in terms of the coordinates x^μ on $\mathcal{N}$ discussed in chapter 2.1, it is easy to see that this invariant contracts to the familiar quantity commonly employed in Minkowski space. Take the following specific mapping

$$y(x) = \begin{cases} y^k = x^k, \ k = 1 \ldots d \\ y^4 = \sqrt{R^2 + r^2}\sin(t/R) \\ y^5 = \sqrt{R^2 + r^2}\cos(t/R) \end{cases} \tag{2.39}$$

($r^2 \stackrel{\text{def}}{=} \vec{x}^2 \stackrel{\text{def}}{=} \sum_{k=1}^{d}(x^k)^2$ and $t \stackrel{\text{def}}{=} x^4$). Using this definition, the metric tensor (2.15) reduces to (2.3) with $\Lambda = -(d-1)R^{-2}$. Equation (2.37) can be rewritten as

$$z(x_1, x_2) = \sqrt{1 + \frac{r_1^2}{R^2}}\sqrt{1 + \frac{r_2^2}{R^2}}\cos\left(\frac{t_1}{R} - \frac{t_2}{R}\right) - \frac{\vec{x_1} \cdot \vec{x_2}}{R^2} \tag{2.40}$$

and in the $R \to \infty$ contraction λ becomes

$$\begin{aligned} &\lim_{R\to\infty} R^2 \lambda(y_1(x_1), y_2(x_2)) \\ &= \lim_{R\to\infty} \left[-(\vec{x_1} - \vec{x_2})^2 + (t_1 - t_2)^2 + + \ldots\right] \\ &= (x_1^\mu - x_2^\mu)(x_{1\mu} - x_{2\mu}). \end{aligned}$$

This makes λ, not unexpectedly, a natural counterpart in anti-de Sitter space of the traditional two-point invariant.

The above expression (2.40) for z becomes particularly transparent when a reference point $y_0 = y(0) = (\vec{0}, 0, R)$ corresponding to the origin in $\mathcal{N}$ is chosen:

$$z(y, y_0) = \sqrt{1 + r^2/R^2}\cos(t/R) \tag{2.41}$$

$$\lambda(y, y_0) = \sin^2(t/R) - (r^2/R^2)\cos^2(t/R) \tag{2.42}$$

This simplification can be put to good use. Computing manifestly $SO(d-1, 2)$ invariant expressions, nothing bars one from choosing a fixed reference point and transforming the result back to the general case afterwards.

2.5 The adS covering space

Figure 2.2 showed how the map (2.39) relating coordinates $x \in \mathcal{N}$ to those in the embedding space $\mathcal{M}$ can be visualized, if two out of three space dimensions are suppressed, as a two-dimensional hyperboloid embedded in three-dimensional space. One peculiar feature of this embedded space, anti-

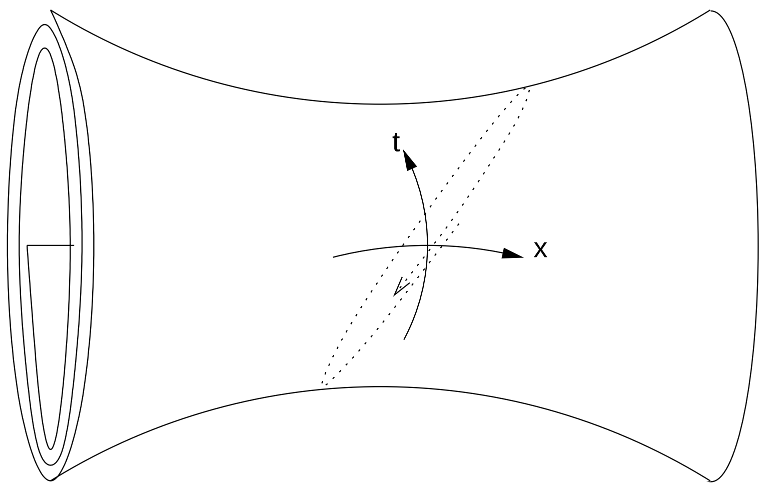


figure 2.3: The universal covering of anti-de Sitter space.

de Sitter space (adS), is immediately evident from both the figure and equation (2.39): it admits closed timelike curves. Actually, it can be shown that any space with a purely timelike boundary at spatial infinity necessarily admits closed timelike curves [Ger68]. Although some relatively recent research suggests this may not be a priori pathological on a microscopic scale [Haw95, FPS92a, for example] and can make for interesting science fiction on a macroscopic one, they would lead to some undesirable physics in the present case, such as a problematic causal structure and quantization of mass giving problems in the Inönü-Wigner contraction to Minkowski space[2]

[2] Some features are already apparent in an S_1 toy model. Consider the wave equation $(\Box + m'^2)\Phi(\tau) = 0$ on one compact dimension, which can be thought of as the circle of radius R. The solutions

$$\Phi_m(\tau) = e^{\pm i m' \tau} \text{ where } m'R \in \mathbb{N}$$

exhibit a discrete rest mass spectrum. This spectrum becomes increasingly dense in the $R \to \infty$ limit to the wave equation on $\mathbb{R}$ but it is clearly impossible to choose m' so that it will approach a given $m \in \mathbb{R}^+$ in a continuous limit.

Going to the covering space $\bar{S}_1 = \mathbb{R}$ by allowing for a cut at $\tau = \pm\pi R$ and introducing a winding number $k \in \mathbb{Z}$ labelling the sheets, the mass spectrum becomes continuous and

(and its reverse, the generalization of physics to anti-de Sitter space). The common and indeed natural way out of this loophole is to extend adS with its $S^1 \times \mathbb{R}^{d-1}$ topology to its $\mathbb{R}^d$ covering space CadS [HE73, Fro74, DvB85]. As shown in figure 2.3, the embedding can now be visualized as a tightly wound scroll. It is evident that the timelike geodesic from the previous picture is no longer closed; it still ends up at the same point in $\mathcal{M}$, but on a different sheet[3].

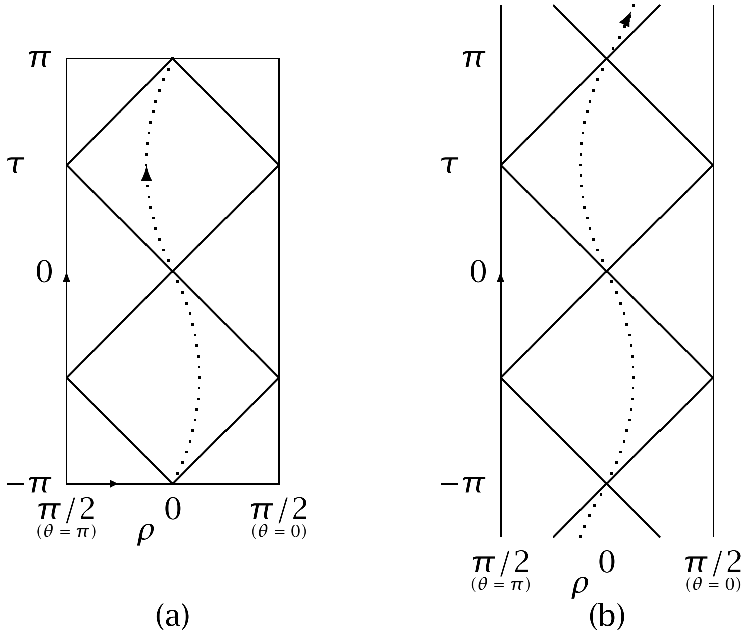


figure 2.4: Penrose diagrams for (a) anti-de Sitter space (adS, top and bottom surfaces are identified) and (b) its covering space (CadS). The dotted line traces a timelike geodesic path.

Although less obvious from figures 2.2 and 2.3, another problem besets adS as well as its covering space: the lack of global hyperbolicity. Consider

---

Φ acquires a k-dependent phase:

$$\bar{\Phi}^k_m(\tau) = e^{2\pi i m' R k}\Phi_m(\tau) \text{ where } m'R \in \mathbb{R}^+$$

Although the $R \to \infty$ limit is now completely trivial – after all, $\bar{S_1} = \mathbb{R}$ even for finite R – the procedure is remarkably similar to that for the anti-de Sitter scalar propagator discussed in section 3.1.

[3]CadS can be thought to be embedded in $\bar{\mathcal{M}} = \mathbb{E}(d-1,2) \otimes \mathbb{Z}$, the covering of $\mathcal{M}$ where $k \in \mathbb{Z}$ counts the windings around the origin in the (y^4, y^5) plane.

the following conformal reparameterization of the coordinates y on (C)adS:

$$\begin{aligned} y(x) &= \begin{cases} y^k = R\tan\rho\ \Omega^k(\theta,\ldots) \\ y^4 = R\sin\tau\,/\cos\rho \\ y^5 = R\cos\tau\,/\cos\rho \end{cases} \text{where} \begin{cases} \tau\in(-\pi,\pi] & \text{(adS)} \\ \tau\in(-\infty,\infty) & \text{(CadS)} \\ \rho\in[0,\pi/2) \end{cases} \\ \mathrm{d}s^2 &= R^2\cos^{-2}\rho\left[\mathrm{d}\tau^2-\mathrm{d}\rho^2-\sin^2\rho\ \mathrm{d}\Omega^2\right] \end{aligned} \tag{2.43}$$

This parameterization, suppressing the angular coordinates Ω, can be drawn in the Penrose diagrams of fig. 2.4. The boundary at spatial infinity $\rho=\pi/2$ is timelike even for CadS, so that information is generally not conserved. The topology of the adS boundary is $S^1\times S^{d-2}$, an extremely interesting topological manifold [Ros94] which gives rise to, among other things, the $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$ singleton fields discussed in section 2.2. SO$(d-1,2)$plays the role of the conformal group on this boundary. In the flat space limit this manifold becomes $\mathbb{R}\times S^{d-2}$, explaining why these singletons necessarily vanish in the contraction. It is also the boundary membrane in bag models, through which confined particles interact with the outside world.

Lack of information conservation makes the space not globally hyperbolic. The Cauchy problem is generally ill-defined. Avis, Isham and Storey [AIS78], noting that anti-de Sitter space with the conformal coordinates above maps into half the $\mathbb{R}\times S^3$ Einstein static universe, and comparing this with the well-known problem of quantizing in a box in Minkowski space, showed that by imposing special boundary conditions and Cauchy data these problems could be avoided and a consistent Hilbert space $\mathcal{H}$ constructed:

- "Transparent" boundary conditions dictate that (massless) fields be C^∞ under the ESU mapping, and specify Cauchy data on the hypersurfaces $\Sigma_1=\{\tau=0,\rho<\pi/2\}$ and $\Sigma_2=\{\tau=\pi,\rho<\pi/2\}$. Conservation laws hold only for two hypersurfaces at τ and $\tau+\pi$ *combined*: information lost to infinity is recycled, a rather unusual if consistent type of conservation.

- "Reflective" conditions demand that, for a field ψ that may be either massless or massive,

$$\lim_{\rho\to\pi/2}\frac{\partial}{\partial\rho}(\cos\rho)^{-1}\psi^a(y)=0 \quad \text{for} \quad \psi^a(\bar{y})=-\psi^a(y) \tag{2.44}$$

$$\lim_{\rho\to\pi/2}(\cos\rho)^{-1}\psi^s(y)=0 \quad \text{for} \quad \psi^s(\bar{y})=+\psi^s(y) \tag{2.45}$$

  where $\bar{y}$ denotes the antipodal point of y (spatial reflection plus a time translation Δt by π/R). The Hilbert space is a direct sum $\mathcal{H}^a\oplus\mathcal{H}^s$.

Cauchy data is specified on a single spatial hypersurface Σ. Spatial infinity acts here as a reflecting wall.

It is also clear from fig. 2.3 and the Penrose diagrams that even in CadS the path of a particle describing a timelike geodesic is periodic with period $\Delta\tau = 2\pi \Rightarrow \Delta t = 1/R$. It resembles that of a relativistic harmonic oscillator [DvB83, Dul84, NNS94]. In the non-relativistic limit $R \to Rc$, $c \to \infty$, the harmonic oscillator is recovered:

$$H = \frac{p^2}{2m} + \frac{mq^2}{2R^2}$$

This peculiarity is used in a number of different bag models. In de Sitter space, on the other hand, the coupling constant changes sign, which is directly related to the absence of a lower bound on the $\mathrm{SO}(d,1)$ energy spectrum.

The two-point invariant revisited

Promoting adS to CadS is a global topological change. Since most of the expressions from the previous sections deal with local quantities only, they remain valid. The two-point invariant (2.36), however, merits some attention at this point. Its behavior in the covering space CadS is most conveniently analyzed in the form (2.42) for a fixed reference point y_0:

$$\lambda(y, y_0) = \sin^2(t/R) - r^2/R^2 \cos^2(t/R).$$

This expression is periodic in t with period πR. Functions of λ will be seen to acquire a phase change for every complete period. The periods will be counted by a sheet number k and the phase will be a function of k.

Consider the function $f_a(y, y_0) = (-\lambda(y, y_0))^a F(\lambda^{-1})$ defined on anti-de Sitter space. F can be a hypergeometrical function, or any other integral power series in λ analytical for $|\lambda| > 1$; the function f_a as a whole is analytical with possible singular points in 1, 0 and ∞. For $a \notin \mathbb{Z}$, this is no longer true: there is a cut at $t = \pm\pi R$. Expressing y is in terms of the coordinates $(t, \vec{r})$ on $\mathcal{N}$, the analytical continuation of $f_a(y(t, \vec{r}), y_0)$ to $t \in \mathbb{R}$ is a function $\bar{f}_a$ on the covering space CadS.

In order to proceed evaluating the phase factor, some assumptions are necessary which will only be fully justified in subsequent chapters (see section 3.1). Expressions like the above typically appear in propagator expressions. To define the integration paths relative to the $\lambda = 0$ singularity, one applies an infinitesimal transformation, similar to the familiar infinitesimal time translation in Minkowski space, leaving the embedding constraint (2.25), $\mathcal{D}^2$ and M^2 (2.34) invariant. From (2.7) the translation's

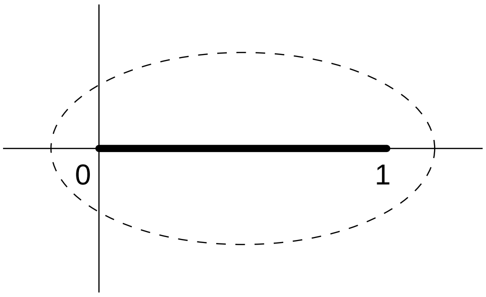


figure 2.5: The path traced through $\mathbb{C}$ by the two-point invariant $\lambda_{\pm\epsilon}$.

adS analogue appears to be generated by M_{54}, the rotation operator in the (y^4, y^5) plane:

$$t \to t \pm i\epsilon R/2 \Rightarrow y \to e^{\pm\frac{\epsilon}{2}M_{54}}y. \tag{2.46}$$

In the specific coordinate map of (2.39)

$$\Rightarrow \begin{cases} \vec{y} & \to & \vec{y} \\ y^4 & \to & y^4\cosh(\epsilon/2) \pm iy^5\sinh(\epsilon/2) \\ y^5 & \to & \mp iy^4\sinh(\epsilon/2) + y^5\cosh(\epsilon/2) \end{cases} \tag{2.47}$$

this can be seen to transform λ to $\lambda_{\pm\epsilon}$ as follows:

$$\begin{aligned} \lambda & \to \lambda_{\pm\epsilon}(y, y_0) \\ & = \lambda\cosh^2(\epsilon/2) \pm i\hat{y}^4\hat{y}^5\sinh(\epsilon) - \frac{(R\hat{y}^5)^2 - (r\hat{y}^4)^2}{R^2 + r^2}\sinh^2(\epsilon/2) \\ & = \lambda \pm i\epsilon\hat{y}^4\hat{y}^5 - \epsilon^2\frac{(\hat{y}^5)^2 - (\hat{y}^4)^2}{4} + \mathcal{O}(\epsilon^3) \\ & = \sin^2(t/R) - \frac{r^2}{R^2}\cos^2(t/R) \pm \frac{i}{2}\epsilon\frac{R^2 + r^2}{R^2}\sin(2t/R) \\ & \quad - \frac{1}{4}\epsilon^2\frac{R^2 + r^2}{R^2}\cos(2t/R) + \mathcal{O}(\epsilon^3) \end{aligned} \tag{2.48}$$

This transformation indeed satisfies the conditions mentioned above. From (2.48) it can be seen that $\lambda_{\pm\epsilon}(t, \vec{r})$ will describe ellipses around 0 and 1 in the complex plane, as shown in figure 2.5; clockwise for $\lambda_{+\epsilon}$, counterclockwise for $\lambda_{-\epsilon}$.

Because of the analyticity of $\bar{f}_a(\lambda(y(t, \vec{r}), y_0))$, the phase is independent of the contour as long as it does not cross the points 0 and 1. Deforming it to a circle $\sim e^{\pm i2t/R}$, it is apparent that the phase $\bar{f}_a$ will pick up for every full revolution $\Delta t = \pi R$ must be $e^{\pm 2\pi ia}$:

$$\bar{f}_a(y; x) = e^{\pm 2\pi iak}(-\lambda(y; x))^a F(\lambda^{-1}(y; x)) = e^{\pm 2\pi iak} f_a(y; x) \tag{2.49}$$

where k is the sheet number. This will be the mechanism whereby propagators evaluated in anti-de Sitter space will be analytically continued to the covering space in section 3.

chapter 3

Propagators

With the enduring interest for four-dimensional anti-de Sitter space, propagators for all types of fields have been the subject of a great many papers, few of which can be named here. Explicit expressions for the homogeneous scalar field propagator have been calculated by Fronsdal [Fro74] and for the inhomogeneous one by Dullemond and van Beveren [DvB85]. Fronsdal and Haugen [FH75] found expressions for the homogeneous spinor propagator, while Janssen and Dullemond [JD86] constructed the spinor Feynman propagator from the scalar one, a procedure that can be iteratively expanded to arbitrary half-integer spin [Les88]. Homogeneous propagators for the electromagnetic field have been derived for a particular gauge by Binegar, Fronsdal and Heidenreich [BFH83] and later for arbitrary gauge by Gazeau [Gaz85]. Janssen and Dullemond [JD87] calculated the Feynman propagator for massive vector fields.

In flat space quantum field theory, diagrams are usually evaluated in momentum space because this tends to greatly simplify computations. However, in curved space-time this becomes a lot less straightforward, as will be discussed in more detail in the introduction to chapter 4. For this reason, configuration space methods are often employed. Unfortunately, the expressions for anti-de Sitter propagators in configuration space will turn out to be fairly complex and hard to do calculations with.

Configuration-space expressions will be derived for the scalar, spinor and vector propagators in d-dimensional anti-de Sitter space and its covering. This is generally well-charted territory and the results represent a modest reworking and extension of the published results quoted above. The expressions derived will be the starting point of a renormalization programme for quantum electrodynamics using the technique of dimensional regularization.

3.1 The CadS scalar propagator

Explicit expressions for scalar two-point functions in four-dimensional anti-de Sitter space and its covering have been calculated by many different authors in as many ways. Fronsdal [Fro74] has found expressions for the homogeneous propagators. Dullemond and van Beveren [DvB85] have used configuration-space methods to obtain the inhomogeneous propagator in a straightforward way. The present discussion will follow the latter, incorporating a slight extension to d-dimensional space.

Although it is textbook material [Col84], it is still worth briefly going through the configuration space formulation of the scalar propagator in Minkowski space for the close parallels it will offer with the treatment below. The scalar Feynman propagator G_F in $\mathbb{E}(d-1,1)$ is a solution to the equation

$$[\Box_2 + m^2]G_F(s(x_2;x_1)) = \delta^d(x_2 - x_1), \tag{3.1}$$

satisfying appropriate boundary conditions, where $s(x_2;x_1) \stackrel{\text{def}}{=} -(x_2 - x_1)^2$. The homogeneous equation valid for $x_1 \neq x_2$ is, in terms of s,

$$\left[4s\frac{\mathrm{d}^2}{\mathrm{d}s^2} + 2d\frac{\mathrm{d}}{\mathrm{d}s} - m^2\right]G(s) = 0.$$

This is essentially a modified Bessel equation; inserting a power series $G(s) = \sum_k a_k s^{\alpha+k}$ gives two solutions, one for $\alpha = 0$ and one for $\alpha = (2-d)/2$. The latter is singular in $s = 0$ and therefore the most interesting starting point for the construction of the Feynman propagator G_F:

$$G_2(s) = \frac{1}{s^{\frac{d-2}{4}}}K_{\frac{d-2}{2}}(ms^{\frac{1}{2}}) = \frac{2^{\frac{d-4}{2}}\Gamma\left(\frac{d-2}{2}\right)}{m^{\frac{d-2}{2}}}\frac{1}{s^{\frac{d-2}{2}}} + \ldots$$

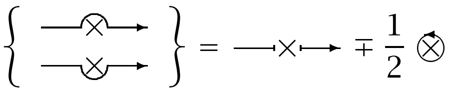
([AS84] 9.6). For the δ^d function to appear at the right hand side of (3.1), together with the right boundary conditions, this solution will have to be modified so as to define the path around this singularity[1] and normalized.

A full treatment would needlessly duplicate Dullemond and van Beveren [DvB85] and the discussion at the end of this chapter; suffice to say that starting with the Ansatz of an infinitesimal time translation $t \to t \pm i\epsilon/2$ and normalizing,

$$G(s_{\pm\epsilon}) = -\frac{2m^{\frac{d-2}{2}}}{2^d\pi^{\frac{d}{2}}}\frac{1}{(s_{\pm\epsilon})^{\frac{d-2}{4}}}K_{\frac{d-2}{2}}(m(s_{\pm\epsilon})^{\frac{1}{2}}) \tag{3.2}$$

[1] This point appears to be glossed over in Collins [Col84] 11.1, who erroneously states that G_F *"is a linear combination of these* [homogeneous] *solutions."* The results of subsequent computations remain unaffected by this issue, fortunately.

then splitting the resultant contours in the principal part and a pure imaginary closed path



and taking things from there, the contour around the pole(s) in the origin finally becomes

which amounts to modifying the above solution to read

$$
\begin{aligned}
G_F(s) &= i\frac{m^{\frac{d-2}{2}}}{2^d\pi^{\frac{d}{2}}}\frac{1}{(s+i\epsilon)^{\frac{d-2}{4}}}K_{\frac{d-2}{2}}(m(s+i\epsilon)^{\frac{1}{2}}) \\
&= i\frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}\left[\frac{1}{(s+i\epsilon)^{\frac{d-2}{2}}}+\dots\right]
\end{aligned}
\tag{3.3}
$$

where the higher order terms in all vanish for the massless $m = 0$ case.

The wave equation

The most general Lagrangian for a free scalar field in curved space-time is [BD82]

$$
\mathcal{L}(x) = \frac{1}{2}[-g(x)]^{\frac{1}{2}}\left\{g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} - [m_0^2 + \xi\mathcal{R}(x)]\phi^2(x)\right\} \tag{3.4}
$$

where $g(x) \stackrel{\text{def}}{=} \det(g_{\mu\nu}(x))$, m_0 is the mass of the field quanta, $\mathcal{R}(x)$ is the Ricci scalar and ξ a possible (dimensionless) coupling with the curvature. Through the action principle the field equation is obtained:

$$
\left[\mathcal{D}^2 + m_0^2 + \xi\mathcal{R}(x)\right]\phi(x) = 0 \tag{3.5}
$$

where

$$
\mathcal{D}^2 \stackrel{\text{def}}{=} g^{\mu\nu}(x)\mathcal{D}_\mu\mathcal{D}_\nu = [-g(x)]^{-\frac{1}{2}}\partial_\mu[-g(x)]^{\frac{1}{2}}g^{\mu\nu}(x)\partial_\nu
$$

Of course, in the case of Anti-de Sitter space, $\mathcal{R}(x) = \mathcal{R} = 2d\Lambda/(d-2)$ is constant everywhere, therefore the gravitational term $\xi\mathcal{R}$ can be absorbed in the definition of the mass m_0 without any more ado. The price to be paid for choosing this so-called minimal coupling is that the massless (conformally invariant) physics does not correspond to the $m_0 = 0$ case, but rather $m_0^2 = -\frac{d(d-2)}{4R^2}$, i.e. an imaginary value of m_0. For the flat space limit $R \to \infty$, though, a reassuring $m_0 = 0$ is recovered for the massless case.

Embedding this equation in $\mathbb{E}(d-1,2)$ as described in chapter 2.4 gives

$$\begin{aligned}
0 &= \left[g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu + m_0^2\right]\phi(x) \\
&= \left[\eta^{MN}y_{M}{}^{,\mu}y_{N}{}^{,\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}(x(y))\partial_\rho) + m_0^2\right]\phi(x(y)) \\
&= \left[\mathcal{D}^M\mathcal{D}_M - (\mathcal{D}^M y_{M}{}^{,\mu})\partial_\mu - y_{M}{}^{,\mu}(\mathcal{D}^M y^K{}_{,\mu})\mathcal{D}_K + m_0^2\right]\phi(x(y)) \\
&= \left[\mathcal{D}^M\mathcal{D}_M + m_0^2\right]\phi(x(y)) \qquad (3.6)
\end{aligned}$$

This can be reduced to the equation for a massless scalar field in $\mathcal{M}$ if $\phi(x)$ is extended to the entirety of $\mathcal{M} = \mathbb{E}(d-1,2)$ in the following manner:

$$\partial_M\partial^M\psi(y) \stackrel{\text{def}}{=} \Box_y\psi(y) = 0, \qquad (3.7)$$

$$\text{where} \qquad \psi(y) \stackrel{\text{def}}{=} \rho^m\phi(y) \stackrel{\text{def}}{=} \rho^m\phi(x(\hat{y}))$$

so that $\phi(y)$ in a neighborhood of $\mathcal{N}$ is a function of $\hat{y}$ – the coordinates y projected on $\mathcal{N}$ – only, and $\psi(y)$ is isotropic in $\rho = |y|$ with degree m. The wave equation for $\phi(y)$ then can be taken to be a wave equation on $\{\mathcal{N} : \rho = R\}$ and is

$$\left[\Box_y + m(m+d-1)R^{-2}\right]\phi(y) = 0 \qquad (3.8)$$

or, using (2.34),

$$\left[M^2 - m(m+d-1)\right]\phi(y) = 0 \qquad (3.9)$$

which is equivalent to (3.6) if one identifies $m_0^2 \stackrel{\text{def}}{=} m(m+d-1)R^{-2}$. The eigenvalue equation (2.10) for the Casimir operator C_1 defined in (2.9) is recovered if $m \stackrel{\text{def}}{=} E_0 - d + 1$.

The conformally invariant massless scalar field equation is recovered for

$$m(m+d-1) = -\frac{d(d-2)}{4} \Leftrightarrow \begin{cases} m = \frac{2-d}{2} \\ m = -\frac{d}{2} \end{cases}$$

Note that the expressions above are invariant under the exchange $-m \leftrightarrow m+d-1$. This symmetry corresponds with the $-E_0 \leftrightarrow E_0 - d + 1$ symmetry in the Casimir operators (2.10) and will return many times in the following chapters.

The homogeneous two point function

The full Feynman propagator $G_F(y_1; y_2)$ can be expected to be invariant under $SO(d-1,2)$, in other words, to be a function of $\lambda(y_1; y_2)$ (2.36). G_F satisfies the field equation (3.8) with a non-homogeneous term at the right hand side. Elementary calculus will show this equation to be a hypergeometrical equation in λ:

$$\begin{aligned} &\left[M^2_{y_2} - m(m+d-1)\right] G_F(\lambda(y_2(x_2); y_1(x_1))) \\ &\quad = -R^2[-g(y_2(x_2))]^{-\frac{1}{2}}\delta^d(x_2 - x_1) \\ \Leftrightarrow \quad & \left[\lambda(1-\lambda)\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} + (\frac{d}{2} - \frac{d+1}{2}\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{1}{4}m(m+d-1)\right] \\ &\quad \times G_F(\lambda(y_2(x_2); y_1(x_1))) = \frac{R^2}{4}\delta^d(x_2 - x_1) \end{aligned} \tag{3.10}$$

In order to solve this, the solution for the homogeneous equation, that is, for $x_1 \neq x_2 \Leftrightarrow y_1 \neq y_2$, is examined first and later these solutions are modified in such a way that the δ^d function emerges. Note that this source term does not simply become $\delta^{d+1}(y)$.

There are two linearly independent solutions, which are transformed into each other by the $-m \leftrightarrow m+d-1$ symmetry. For $m > -d+1$, the solution well-behaved for $\lambda \to -\infty$ is[2]

$$\phi_{m>-d+1}(\lambda) \quad = \quad (-\lambda)^{-\frac{m+d-1}{2}} F\left(\tfrac{m+d-1}{2}, \tfrac{m+1}{2}; m + \tfrac{d+1}{2}; \tfrac{1}{\lambda}\right) \tag{3.11}$$

In this m domain, the other solution is unphysical because of its bad $\lambda \to -\infty$ behavior — this amounts to imposing boundary conditions at infinity [AIS78, DvB85]. The $m < 0$ expression follows from simple substitution; in the following, attention will be limited to $m > -d+1$ and the domain label suppressed.

The present solution's analytical continuation to $|\lambda| < 1$ is, up to a constant factor [BMP53],

$$\begin{aligned} \phi(\lambda) \quad = \quad & 2^{2-d}\Gamma\left(\tfrac{2-d}{2}\right)\frac{\Gamma(m+d-1)}{\Gamma(m+1)} F\left(\tfrac{m+d-1}{2}, -\tfrac{m}{2}; \tfrac{d}{2}; \lambda\right) \\ & + (-\lambda)^{-\frac{d-2}{2}}\Gamma\left(\tfrac{d-2}{2}\right) F\left(\tfrac{m+1}{2}, -\tfrac{m+d-2}{2}; \tfrac{4-d}{2}; \lambda\right) \end{aligned} \tag{3.12}$$

The first term has a pole for $d \to 4$ in $\Gamma\left(\frac{2-d}{2}\right)$. The second also has a simple pole due to $F(a, b; \frac{4-d}{2}; \lambda)$. These poles cancel, so that ϕ as a whole is regular for $d \to 4$. However, ϕ has a singularity in $\lambda = 0$ as well:

[2] This is of course by no means the only solution satisfying this equation, but it is the one satisfying the symmetry properties expected of homogeneous propagators. See [Fro74] for the general P_4, $\vec{J}^2$ and J_3 (see section 2.2) eigenstate function at $d = 4$.

- $d \notin \mathbb{N}$: a $(-\lambda)^{-\frac{d-2}{2}}$ divergence;
- $d \to 4$: a $\lambda^{-1} + \ln(-\lambda) f(\lambda)$ divergence, where f is regular[3]. The logarithmic divergence comes in from the canceling $d \to 4$ poles discussed above. Note that the propagator has the Hadamard form in this limit.

In order to gain insight in graph divergences later on, it is a useful exercise to extract the singularities. Following Collins [Col84], the expression can be split in a regular $\mathcal{O}(\lambda^0)$ and a singular $\mathcal{O}(\lambda^{-\frac{d-2}{2}})$ part as follows:

$$\begin{aligned}\phi^{(\text{sing})}(\lambda) &= (-\lambda)^{-\frac{d-2}{2}} \Gamma\left(\tfrac{d-2}{2}\right) F\left(\tfrac{m+1}{2}, -\tfrac{m+d-2}{2}; \tfrac{4-d}{2}; \lambda\right) \\ &\quad -\frac{1}{2}\frac{\mu^{d-4}}{4-d}(\pi R^2)^{\frac{d-4}{2}}(m+1)(m+2) F\left(\tfrac{m+3}{2}, -\tfrac{m}{2}; 2; \lambda\right) \qquad (3.13)\end{aligned}$$

$$\begin{aligned}\phi^{(\text{ana})}(\lambda) &= -iR^{2-d}(4\pi)^{-\frac{d}{2}} \\ &\quad \cdot \left\{ \frac{\Gamma\left(\frac{2-d}{2}\right)\Gamma(m+d-1)}{\Gamma(m+1)} F\left(\tfrac{m+d-1}{2}, -\tfrac{m}{2}; \tfrac{d}{2}; \lambda\right) \right. \\ &\qquad \left. + \frac{2\mu^{d-4}}{4-d}(4\pi R^2)^{\frac{d-4}{2}}(m+1)(m+2) F\left(\tfrac{m+3}{2}, -\tfrac{m}{2}; 2; \lambda\right) \right\} \qquad (3.14)\end{aligned}$$

The μ scale factor has been inserted to obtain the correct dimensionality for $d \neq 4$ in all terms. Note that a naive approach would have resulted in two parts both singular in the $d \to 4$ limit.

Working out the full expressions (see Appendix A.1) it can be observed that neither ϕ nor its derivatives contain pole terms in λ exhibiting a $d \to 4$ singularity. Had this been the case, the two types of singularities would have been inextricably mixed, making it impossible to separate out a part regular in both λ and d. This would break the procedure followed later on.

The inhomogeneous two point function

The two point function presented in the preceding section is, of course, not the Feynman propagator $iR^2 G_F(\lambda) = \langle 0|T\phi(y_2)\phi^\dagger(y_1)|0\rangle$ proper. The latter should be an analytical solution of the full inhomogeneous equation (3.10) and even in t. Fortunately, the two point function presented so

[3] The full $d \to 4$ expression can be found using Bateman [BMP53]

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\frac{m+1}{2})_{n+1}(-\frac{m+2}{2})_{n+1}}{2n!(n+1)!}\lambda^n \Big[&-\ln(-\lambda) + \psi(n+2) + \psi(n+1) \\ &-\psi(\tfrac{m+2}{2} - n) - \psi(\tfrac{m+3}{2} + n)\Big] + \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\lambda}\end{aligned}$$

which agrees with [DvB85].

far already contains all the necessary ingredients; indeed, it *is* the Feynman propagator minus some small modifications dictating the right path when it comes to evaluating integrals.

Using Gauss' theorem to reduce the integration to one over the spatial part of $\mathcal{N}$, and the fact that G_F gives no surface terms in spatial infinity, equation (3.10) means that

$$\int_{\Sigma_{\pm a}} \mathrm{d}^{d-1}\vec{y}\ \frac{\partial}{\partial y^4} G_F(\lambda(y;x))\bigg|_{y^4-x^4=\pm a} = \operatorname{sgn}(a). \tag{3.15}$$

Here, y^4 is actually a convenient notation for the dth coordinate and $\vec{y}$ are the spatial components $(y^1,\dots,y^{d-1})$ in (2.39). G_F will now be constructed by working towards this result, starting with the two-point function ϕ as the main ingredient. The construction presented here follows the procedure outlined for the $d = 4$ case by Dullemond and van Beveren [DvB85].

First of all, in much the same way as in Minkowski space, it is necessary to define the t integration paths (see section B.2) around ϕ's $\lambda = 0$ singularity by performing an imaginary translation in time, which in anti-de Sitter space becomes a rotation in the (y^4, y^5) plane (2.47):

$$\lambda_{\pm\epsilon}(y;y_0) = \lambda \pm i\epsilon\hat{y}^4\hat{y}^5 - \epsilon^2\frac{(\hat{y}^5)^2 - (\hat{y}^4)^2}{4} + \mathcal{O}(\epsilon^3).$$

This amounts to defining integration paths to be  for $\phi(\lambda_{+\epsilon})$ and  for $\phi(\lambda_{-\epsilon})$, respectively. Now consider the integration in (3.15) performed for the homogeneous $\phi(\lambda_{\pm\epsilon})$ as given by (3.12). It can be shown that only the highest order divergence in ϕ contributes: starting with an arbitrary power of λ

$$\begin{aligned}
&\int_{\Sigma_0} \mathrm{d}^{d-1}\vec{y}\ \frac{\partial}{\partial y^4}(-\lambda_{\pm\epsilon}(y;x))^{-k}\bigg|_{y^4=x^4} \\
&= \int_{\Sigma_0'} \mathrm{d}^{d-1}\vec{y}\ \frac{\partial}{\partial y^4}(-\lambda_{\pm\epsilon}(y;y_0))^{-k}\bigg|_{y^4=0} \\
&= \pm i\epsilon\frac{k}{R}\int \mathrm{d}^{d-2}\Omega \int \mathrm{d}r \frac{r^{d-2}\sqrt{1+r^2/R^2}}{(r^2/R^2(1+\epsilon^2/4)+\epsilon^2/4)^{k+1}}
\end{aligned}$$

the substitution $r^2/R^2 \to \rho\cdot\epsilon^2/(\epsilon^2+4)$ can be made, the square root expanded into a series in $\sum_l a_l r^l$ and the integration evaluated ([AS84] 6.2.1), resulting in a sum

$$= \pm i\sum_l a_l\epsilon^{l+d-2k-2}\frac{2^{2k+1}R^{d+l-2}k}{(\epsilon^2+4)^{\frac{d-1+l}{2}}}\Omega_{d-1}B\Big(\frac{d+l-1}{2},\frac{1-l}{2}\Big).$$

The most divergent term in the scalar propagator (3.12) is $\lambda^{\frac{2-d}{2}}$, giving $k = \frac{d-2}{2}$ which makes the terms in the sum proportional to ϵ^l. For vanishing ϵ, then, only the lowest order $l = 0$ term in the expansion of $\sqrt{1 + r^2/R^2}$ survives:

$$\overset{k=\frac{d-2}{2}}{=} \pm 2\pi^{\frac{d}{2}} i \frac{R^{d-2}}{\Gamma\left(\frac{d-2}{2}\right)} + \mathcal{O}(\epsilon).$$

The higher order terms of ϕ yield higher powers of ϵ in the expressions above and will all vanish in the $\epsilon \to 0$ limit. The (3.15) integral for ϕ is therefore determined by the most singular term only.

$$\int\limits_{\Sigma_0} \mathrm{d}^{d-1}\vec{y}\, \frac{\partial}{\partial y^4} \phi(\lambda_{\pm\epsilon}(y;x)) \bigg|_{y^4 = x^4} = \pm 2\pi^{\frac{d}{2}} i R^{d-2} + \mathcal{O}(\epsilon).$$

Since from (2.48) and (3.12) one can infer $\lim_{\epsilon\to 0} \frac{\partial}{\partial y^4}\phi(\lambda_{\pm\epsilon})|_{y^4=0} = 0$ for $\vec{y} \neq \vec{0}$, it follows that $\lim_{\epsilon\to 0} \frac{\partial}{\partial y^4}\phi(\lambda_{\pm\epsilon})|_{y^4=x^4}$ must in fact be proportional to the delta function of $\vec{y}$:

$$\lim_{\epsilon\to 0} \frac{\partial}{\partial y^4} \phi(\lambda_{\pm\epsilon}(y;x)) \bigg|_{y^4=0} = \pm 2\pi^{\frac{d}{2}} i R^{d-2} \delta^{d-1}(\vec{y} - \vec{x}). \tag{3.16}$$

The (normalized) real and imaginary parts of $\phi(\lambda_{\pm\epsilon})$

$$\frac{1}{2\pi^{\frac{d}{2}} R^{d-2}} \phi(\lambda_{\pm\epsilon}) = \left\{ \begin{array}{c} \longrightarrow\!\!\overset{\frown}{\otimes}\!\!\longrightarrow \\ \longrightarrow\!\!\underset{\smile}{\otimes}\!\!\longrightarrow \end{array} \right\} = \longrightarrow\!\times\!\longrightarrow \mp \frac{1}{2} \otimes = G(\lambda) \pm i\bar{G}(\lambda)$$

where

$$\begin{aligned} G(\lambda) &\overset{\text{def}}{=} \frac{1}{2\pi^{\frac{d}{2}} R^{d-2}} \mathrm{Re}\, \phi(\lambda_{+\epsilon}) = \longrightarrow\!\times\!\longrightarrow \\ \bar{G}(\lambda) &\overset{\text{def}}{=} \frac{1}{2\pi^{\frac{d}{2}} R^{d-2}} \mathrm{Im}\, \phi(\lambda_{+\epsilon}) = -\frac{1}{2i} \otimes \end{aligned} \tag{3.17}$$

are themselves full solutions of the homogeneous wave equation. G is even in $y^4 - x^4$ (and $t_y - t_x$); $\bar{G}$ is odd in $y^4 - x^4$ and vanishes for $y^4 = x^4, \vec{y} \neq \vec{x}$. The imaginary part is the most interesting as it gives rise to the $\delta^{d-1}(\vec{y} - \vec{x})$ in (3.16). In fact, multiplied by $\theta(\pm(y^4 - x^4))$ to select only future or past support

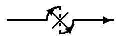
$$G_{\text{ret,adv}}(\lambda(y;x)) = \pm\theta(\pm(t_y - t_x))\bar{G}(\lambda(y;x)) = \left\{ \begin{array}{l} \frac{1}{i} \nearrow\!\!\!\!\times \\ \frac{1}{i} \swarrow\!\!\!\!\times \end{array} \right. \tag{3.18}$$

they satisfy

$$\int_{\Sigma_a} \mathrm{d}^{d-1}\vec{y}\; \frac{\partial}{\partial y^4} G_{\text{ret,adv}}(\lambda(y;x))\bigg|_{y^4-x^4=a} = \pm\theta(\pm a)$$

and can by applying Gauss' theorem over the surfaces $y^4 = \pm a$ to the above expression be seen to represent the retarded and advanced solution, respectively, to the inhomogeneous wave equations (3.10) and (3.15). Finally, the propagator defined by

$$G_F(\lambda(y;x)) = G_{\text{ret}}(\lambda(y;x)) - \frac{1}{2i}\frac{1}{2\pi^{\frac{d}{2}}R^{d-2}}\phi(\lambda_{+\epsilon}(y;x))$$

$= \frac{1}{i}$  $- \frac{1}{2i}$  $= -\frac{1}{2i}$ 

$$= -\frac{1}{2i}\left[G(\lambda(y;x)) - i\,\mathrm{sgn}(t_y - t_x)\bar{G}(\lambda(y;x))\right] \qquad (3.19)$$

can readily be seen to satisfy not only the same wave equations, but also the analyticity and symmetry requirements for the Feynman propagator mentioned above. The path shown can be clarified somewhat by showing the pole $\times$ in two halves corresponding to $t < 0$ and $t > 0$, each with half the residue, giving

 $=$ 

This amounts to the replacement of $\lambda(y;x)$ by $\lambda_F(y;x)$ in the homogeneous solution $\phi(\lambda)$

$$G_F(\lambda(y;x)) = -\frac{1}{2i}\frac{1}{2\pi^{\frac{d}{2}}R^{d-2}}\phi(\lambda_F(y;x)) \qquad (3.20)$$

$$\lambda_F(y;x) \stackrel{\text{def}}{=} \theta(t_y - t_x)\lambda_{-\epsilon}(y;x) + \theta(-t_y + t_x)\lambda_{+\epsilon}(y;x). \qquad (3.21)$$

Expanded, λ_F is only a small modification of $\lambda_{\pm\epsilon}$ (2.47):

$$\lambda_F(y;y_0) = \lambda(y;y_0) - i\epsilon\,\mathrm{sgn}(t)\hat{y}^4\hat{y}^5 - \epsilon^2\frac{(\hat{y}^5)^2 - (\hat{y}^4)^2}{4} + \mathcal{O}(\epsilon^3).$$

Extension to the covering space

The discussion so far has concerned itself with the case where the points of the two point function are relatively close. Globally, fact is that the two point function is living in the covering space CadS rather than plain anti-de Sitter space. This brings about some modifications to the expressions derived above. Fortunately, extending them to the covering space is a straightforward task given the groundwork done in the previous chapter.

Recalling from section 2.5 that the phase factor introduced when moving from sheet to sheet is determined by the (fractional) power of λ, the continuation of ϕ to the covering space is most easily evaluated for $\lambda \to -\infty$ (3.11) where the power is directly exposed: $\phi(-\lambda) \sim (-\lambda)^{-\frac{m+d-1}{2}}$. Using (2.49) it follows that

$$\phi^{(k)}(\lambda_{\pm\epsilon}(y;x)) = e^{\mp\pi i(m+d-1)(k_y-k_x)}\phi(\lambda_{\pm\epsilon}(y;x)).$$

The real and imaginary parts (3.17) become

$$\begin{aligned} G^{(k)}(\lambda) &= \cos[\pi(m+d-1)\Delta k]G(\lambda) - \sin[\pi(m+d-1)\Delta k]\bar{G}(\lambda) \\ \bar{G}^{(k)}(\lambda) &= \sin[\pi(m+d-1)\Delta k]G(\lambda) + \cos[\pi(m+d-1)\Delta k]\bar{G}(\lambda) \end{aligned}$$

Note that for $d \in \mathbb{N}$, mixing only takes place for $m \notin \mathbb{Z}$ and that plain anti-de Sitter space, i.e. no phase factor, would have imposed a severe $\frac{m+d-1}{2} \in \mathbb{Z}$ constraint on the physics.

Extension of the Feynman propagator is almost as straightforward,

$$G_F^{(k)}(\lambda(y;x)) = e^{-\pi i(m+d-1)|k_y-k_x|}G_F(\lambda(y;x)). \tag{3.22}$$

but this implies that the $\lambda = 0$ singularity be present on every sheet. This may seem worrying initially, but seeing that G_F is constructed using $\operatorname{sgn}(t_y - t_x) = \delta_{k0}\operatorname{sgn}(y^4)$, not simply $\operatorname{sgn}(y^4)$, the integration path about the pole in the zeroth sheet (with respect to the reference point y_0, i.e. the sheet where x and y meet in the general case) is unique. Indeed, the extended integration path is

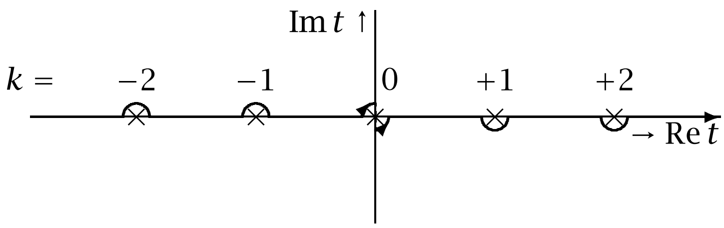


figure 3.1: The Feynman propagator's time integration path in CadS.

and in Appendix B it is shown that as a consequence only the $k = 0$ contribution will be relevant.

3.2 The CadS spinor propagator

Spin-$\frac{1}{2}$ fields in anti-de Sitter space have been studied extensively. In his programme on elementary particles, Fronsdal et al studied massive [FH75] and massless [FF80a] particles. Theirs is by no means the only way to represent a spinning particle in anti-de Sitter space; a few years ago, Kuzenko et al [KLSS96] have described a spin-s particle using a mechanical model with a $SO(3,2) \times S^2$ configuration space, but their model was conceived in the context of superstrings and provides no computational advantages here.

One of the simplest ways to derive spinor wave functions and propagators is to construct them from their scalar counterparts [JD86]; in fact, this construction can then be iteratively extended to a arbitrary massive half-integer spin fields [Les88] by observing that

$$\begin{aligned} &D(m,s) \otimes D(-1,0) = \\ &\quad D(m,s+1) \oplus D(m,s) \oplus D(m,s-1) \oplus D(m-1,s) \oplus D(m+1,s) \end{aligned}$$

(a fact also used by Gazeau and Hans [GH88] in the context of integer-spin fields). Unfortunately, his procedure is quite outside the scope of the present text. After briefly touching upon the well-known procedure in Minkowski space, a slight extension to Janssen's work will be discussed.

The spinor propagator S_F in Minkowski space $\mathbb{E}(d-1,1)$ is a solution to the equation

$$[i\partial\!\!\!/_2 - m]S_F(x_2;x_1) = \delta^d(x_2 - x_1) \tag{3.23}$$

satisfying appropriate boundary conditions. It is a well known fact (for example, Itzykson and Zuber [IZ80] 2-115) that since

$$\begin{aligned} &-[i\partial\!\!\!/_2 - m][i\partial\!\!\!/_2 + m]G_F(x_2 - x_1) \\ &\quad = [\Box_2 + m^2]G_F(x_2 - x_1) = \delta^d(x_2 - x_1) \end{aligned}$$

the spinor propagator can be directly related to the scalar propagator as

$$S_F(x_2 - x_1) = -[i\partial\!\!\!/_2 + m]G_F(x_2 - x_1) \tag{3.24}$$

This construction can be simply extended to anti-de Sitter space, as shown below. In fact, arbitrary massive half-integer spin fields in $\mathrm{SO}(3,2)$ can be constructed [Les88].

The wave equation

The generalization of the higher-spin equations of motions to arbitrary curved space is [BD82]

$$[i\mathcal{D\!\!\!/} - m]\,\psi = 0 \tag{3.25}$$

where the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}{}_\mu D_{ab} \tag{3.26}$$

is determined by the representation D_{ab} of the local symmetry group according to which the field ψ transforms. In the case of the generalized Dirac equation governing spinor fields, D is given by

$$D_{ab} = \sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \tag{3.27}$$

The gamma matrices γ_a and γ_μ are defined by the anticommutator relations

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu},\ \mu,\nu = 1\ldots d; \\ \{\gamma_a, \gamma_b\} &= 2\eta_{ab},\ a,b = 1\ldots d, \end{aligned}$$

where

$$\gamma_\mu = V^a_\mu(x)\gamma_a,$$

so that $\partial\!\!\!/ = \gamma^a\partial_a = \gamma^\mu\partial_\mu$. These relations define a Clifford algebra forming a multiplicative group Γ with 2^{d+1} elements. In the course of the argument, one additional gamma matrix corresponding to the extra dimension in the embedding space will be required; it is defined by extending the above relation for γ_a to $A, B = 1\ldots d, 5$:

$$\{\gamma_A, \gamma_B\} = 2\eta_{AB},\ A,B = 1\ldots d, 5. \tag{3.28}$$

where $\eta_{55} = +1$. Note that the notation γ_5 has been chosen merely for reasons of clarity; for the moment, it is *not* supposed that $d = 4$, much less that $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$. Also, no particular representation can be singled out as yet[4]. Eventually, it shall be assumed that the γ_a become the normal Dirac

[4]The group has $N_\mu = 2^d + 1$ (d even) or $N_\mu = 2^d + 2$ (d odd) classes. Therefore, while for $d = 4$ there is just one non-trivial, faithful, four-dimensional irreducible representation ($N_\mu = 17, n_1 = \ldots = n_{16} = 1, n_{17} = 4, \Sigma n_\mu^2 = N_\Gamma = 32$), for $d = 5$ there actually are *two* such irreps, neither of which is faithful ($N_\mu = 34, n_1 = \ldots = n_{32} = 1, n_{33} = n_{34} = 4, \Sigma n_\mu^2 = N_\Gamma = 64$). Taking the $d \to 4$ limit of Γ is not a trivial procedure. In the present context, however, these considerations have no physical impact and will be mostly side-stepped.

gamma matrices[5] in the limit $d \to 4$; this is not trivial but the mechanism is outside the scope of the present discussion.

The idea is then to rewrite equation (3.25) to one for a field defined on at least a neighborhood around $\mathcal{N}$ in the embedding space $\mathcal{M}$. It is convenient[6] to define a second set of gamma matrices for use in $\mathcal{M}$:

$$\begin{aligned} (\bar{\gamma}_A) &= (i\gamma_5\gamma_a, \gamma_5) \\ \{\bar{\gamma}_A, \bar{\gamma}_B\} &= 2\eta_{AB},\ A, B = 1 \dots d+1 \\ \sigma_{AB} &= \tfrac{1}{4}[\bar{\gamma}_A, \bar{\gamma}_B] \end{aligned} \tag{3.29}$$

Note that $\sigma_{A=a\ B=b} = \sigma_{ab}$; the two definitions are consistent and it is unnecessary to distinguish between them. The matrices σ_{AB} realize the spinor representation of $\mathrm{sp}(d) \sim \mathrm{so}(d-1,2)$. Their exponentiation defines the spinor representation of $\mathrm{Sp}(d)$, the double covering of the anti-de Sitter symmetry group: $\mathrm{SO}(d-1,2)^{\uparrow} \sim \mathrm{Sp}(d)/\mathbb{Z}_2$. In the embedding formalism, a spinor part $S_{MN} = \sigma_{MN}$ is added to the angular momentum operator M_{MN} defined in (2.32):

$$J_{MN} = M_{MN} + S_{MN}.$$

The covariant derivative can now be written as follows:

$$\begin{aligned} \mathcal{D}\!\!\!/\psi &= \left[\partial\!\!\!/ + \frac{1}{2}G^a{}_M(x)\partial\!\!\!/ G^{-1Mb}(x)\sigma_{ab}\right]\psi(x) \\ &= -\bar{\gamma}_5\left[\partial\!\!\!/ + \frac{1}{2}G^A{}_M(x)\partial\!\!\!/ G^{-1MB}(x)\sigma_{AB} - \frac{1}{2}G^a{}_M(x)\partial\!\!\!/ G^{-1M5}(x)\sigma_{a5}\right. \\ &\qquad \left. -\frac{1}{2}G^5{}_M(x)\partial\!\!\!/ G^{-1Mb}(x)\sigma_{5b}\right]\bar{\gamma}_5\psi(x) \\ &= -\bar{\gamma}_5\left[\partial\!\!\!/ + \frac{1}{2}G^A{}_M(x)\partial\!\!\!/ G^{-1MB}(x)\sigma_{AB} + i\frac{d}{2R}\right]\bar{\gamma}_5\psi(x) \\ &= i\left[\left(\bar{\gamma}^K - \bar{\gamma}^5 G^{-1K}{}_5(x)\right)\left(\partial_K + \frac{1}{2}G^A{}_M(x)\partial_K G^{-1MB}(x)\sigma_{AB}\right)\right. \\ &\qquad \left. -\frac{d}{2R}\bar{\gamma}_5\right]\bar{\gamma}_5\psi(x) \end{aligned}$$

[5] Of course, this procedure would be highly suspect in a situation where the γ_5 is used to represent axial currents, as there is no way to recover the usual γ_5 in a continuous $d \to 4$ limit unless either the anticommutation relations are modified or the first four γ's are given a special status. This gives rise to the well-known anomaly for axial currents. However, for embedding purposes there is no "anomalous" effect propagating into the physical result and these intricacies shall be ignored. See also Collins [Col84] 4.6 and 13.2.

[6] In order to avoid unwanted γ_5 terms generated by σ_{a5} while retaining a formulation closely resembling the usual one, the definition of a new set of gamma matrices is necessary for use in $\mathcal{M}$. This definition absorbs awkward i and γ_5 factors which would otherwise crop up.

Here, as in section 2.3, indices $A, B, \ldots$ and indices $K, L, \ldots$ refer to two different bases of $T(y, \mathcal{M})$ related by the vielbein-like transformation $G^A{}_M$ (2.13, 2.28); the $A = 5$ index represents the extra dimension added by the embedding.

To proceed, $\psi(x)$ and $G(x)$ will have to be extended to a neighborhood of $\mathcal{N}$ in $\mathcal{M}$ using some kind of prescription. They will be chosen in such a way that in a neighborhood of $\mathcal{N}$ they are isotropic in $R \stackrel{\text{def}}{=} |y|$, i.e. functions of the projected coordinates $x(\hat{y})$ on $\mathcal{N}$ only: if $\partial_R = G^{-1M}{}_5(x)\partial_M = \hat{y}^M(x)\partial_M$ then

$$\partial_R \psi(\hat{y}) = \partial_R G(\hat{y}) = 0.$$

Using this choice[7], the covariant derivative of the spinor field finally turns out to be

$$\mathcal{D}\psi = i\left[\bar{\partial} + \frac{1}{2}G^A{}_M \bar{\partial} G^{-1MB}\sigma_{AB} - \frac{d}{2R}\bar{\gamma}_5\right]\bar{\gamma}_5\psi, \tag{3.30}$$

where $\bar{\partial} = \bar{\gamma}^M\partial_M = \bar{\gamma}^A G^{-1M}{}_A(\hat{y})\partial_M$.

The next step, naturally, is to cast the Dirac equation (3.25) in the new mould: suppressing the $\hat{y}$ dependency for conciseness,

$$\begin{aligned}[i\mathcal{D} - m]\,\psi &= -\left[\bar{\partial} + \frac{1}{2}G^A{}_M\bar{\partial}G^{-1MB}\sigma_{AB} - \frac{d}{2R}\bar{\gamma}_5 + m\bar{\gamma}_5\right]\bar{\gamma}_5\psi \\ &= -\bar{\gamma}^K\left[\partial_K + \frac{1}{2}G^A{}_M\partial_K G^{-1MB}\sigma_{AB} + R^{-1}(mR - d/2)\hat{y}_K\right]\bar{\gamma}_5\psi \\ &= 0\end{aligned} \tag{3.31}$$

This equation (3.31) can in fact be recognized as that of a spinor field in $\mathcal{M} = \mathbb{E}(d-1, 2)$; the first two terms constitute the covariant derivative

$$\begin{aligned}\mathcal{D}_M &= \partial_M + U\partial_M U^{-1} \\ &= \partial_M - \frac{1}{2}e^{AN}\partial_M e^B{}_N\sigma_{AB}\end{aligned}$$

where the vielbein $e^A{}_M$ can be identified as $G^{-1A}{}_M$, and

$$e^A{}_M\gamma_A = U\gamma_M U^{-1}$$

[7] An alternative approach would be to impose isotropy of order N: $y^M\partial_M\psi = N\psi$. This definition can be used to absorb the mass term of Eqn. (3.25) in such a way that the massive theory in $\mathcal{N}$ corresponds to the conformally invariant (massless) theory in $\mathcal{M}$, in close analogy to the anisotropic $\psi(y)$ satisfying the "massless" (3.7) versus $\phi(\hat{y})$ governed by (3.8). This approach offers no real computational advantages and will not be further explored here.

describes the connection between local Lorentz transformations and SU(2), represented by the vielbein $e^A{}_M$ and the spinor transformation U respectively. The SO$(d-1,2)$ transformation G^{-1} will take this back to the natural vielbein $e^A{}_M = \delta^A_M$ for $\mathbb{E}(d-1,2)$:

$$\begin{aligned}[i\not{D} - m]\,\psi &= -U\bar{\gamma}^A\left[\partial_A + R^{-1}(mR - d/2)\hat{y}_A\right]\hat{\gamma}\psi' \\ &= 0 \end{aligned} \tag{3.32}$$

where $\psi' = U^{-1}\psi$ and $\hat{\gamma} = U^{-1}\bar{\gamma}_5 U$.

The homogeneous two point function

The solution for above equation can be derived from special solutions of the massless scalar field Φ in $\mathcal{M}$. Section 3.1 started out with the massless scalar field equation (3.7)

$$\partial^2\Phi = 0$$

and singling out the solutions with SO$(n, d-n)$ symmetry: $\Phi(y) = \rho^{m'}\phi(\hat{y})$, Eqn. (3.8) was obtained

$$\left[\partial^2 + m'(m' + d - 1)R^{-2}\right]\phi = 0.$$

In order to express the spinor field in terms of the scalar field, the square root of this operator is taken in such a way that the differential equation for the spinor field appears on the left side. Starting from the Ansatz

$$\begin{aligned} 0 &= \bar{\gamma}^A\left[R\partial_A + \alpha\hat{y}_A\right]\bar{\gamma}^B\left[R\partial_B + \beta\hat{y}_B\right]\phi \\ &= \Big[R^2\partial^2 + \beta(\alpha + d) + (\alpha + \beta + 1)y\cdot\partial \\ &\qquad + 2(\alpha - \beta + 1)\sigma^{AB}y_A\partial_B\Big]\phi \\ &\overset{?}{=} R^2\left[\partial^2 + m'(m' + d - 1)R^{-2}\right]\phi \end{aligned}$$

one gets, using $y\cdot\partial\phi = 0$, the two solutions $\alpha = m' - 1, \beta = m'$ or $\alpha = -m' - d, \beta = -m' - d + 1$. Comparison with (3.32) would lead to the identifications $m' = mR - d/2 + 1$ and $m' = -mR - d/2$, respectively. When attention is restricted to the domain $m' > -d + 1$, as was done in the construction of the scalar propagator, the first solution is the one to take:

$$m' = mR + \frac{2-d}{2}. \tag{3.33}$$

With these definitions (3.8) becomes

$$\bar{\gamma}^A\left[R\partial_A + (m' - 1)\hat{y}_A\right]\bar{\gamma}^B\left[R\partial_B + m'\hat{y}_B\right]\phi = 0 \tag{3.34}$$

which yields the following solution to the embedded spinor equation (3.31):

$$\hat{y}\!\!\!/\,\psi' = \bar{y}^B \left[R\partial_B + m'\hat{y}_B \right] \phi \tag{3.35}$$

For this to be a valid solution of (3.31) it must satisfy the homogeneity constraint on $\psi(\hat{y})$ used to arrive at this equation: it must be shown to be independent of R: $\partial_R \psi = 0$. This is a fairly straightforward exercise given $\partial_R \phi = 0$. Re-expressed in terms of quantities in the embedded space $\mathcal{N}$ this relation (3.35) becomes

$$\psi = -\left[i\mathcal{D}\!\!\!/ + R^{-1}(m' + d/2) \right] U\phi \tag{3.36}$$

The inhomogeneous two point function

The last two relations from the previous paragraph will serve to give the spinor propagator $iR^2 S_F(y_2, y_1) = \langle 0|T\psi(y_2)\bar{\psi}(y_1)|0\rangle$. This propagator should satisfy

$$\left[i\mathcal{D}\!\!\!/_{x_2} - m \right] S_F(x_2; x_1) = [-g(x_2)]^{-\frac{1}{2}} \delta^d(x_2 - x_1)$$

or, in $\mathcal{M}$,

$$\begin{aligned} &\bar{y}^A \left[R\partial_{y_2 A} + (m' - 1)\hat{y}_{2A} \right] \hat{y}\!\!\!/ \hat{y}\!\!\!/_2 S_F'(y_2(x_2); y_1(x_1)) \\ &\quad = R^2 [-g(x_2)]^{-\frac{1}{2}} \delta^d(x_2 - x_1) \end{aligned} \tag{3.37}$$

Defining, analogously to (3.35),

$$\hat{y}\!\!\!/ S_F'(y_2; y_1) = \bar{y}^B \left[R\partial_{y_2 B} + m'\hat{y}_{2\ B} \right] G_F(\lambda(y_2; y_1)), \tag{3.38}$$

where $G_F(\lambda(y_2; y_1))$ is the scalar propagator, one finds

$$\begin{aligned} &\bar{y}^A \left[R\partial_{y_2\ A} + (m' - 1)\hat{y}_{2\ A} \right] S_F'(y_2; y_1) \\ &\quad = \bar{y}^A \left[R\partial_{y_2 A} + (m' - 1)\hat{y}_{2\ A} \right] \\ &\qquad \times \bar{y}^B \left[R\partial_{y_2 B} + m'\hat{y}_{2\ B} \right] G_F(\lambda(y_2; y_1)) \\ &\quad = R^2 \left[\partial^2_{y_2} + m'(m' + d - 1) \right] G_F(\lambda(y_2; y_1)) \\ &\quad = R^2 [-g(x_2)]^{-\frac{1}{2}} \delta^d(x_2 - x_1) \end{aligned}$$

so Eqn. (3.38) indeed generates a spinor propagator from the scalar one. Again, this equation may well be carried back to intrinsic coordinates and fields on anti-de Sitter space:

$$\begin{aligned} S_F(x_2; x_1) &= -\left[i\mathcal{D}\!\!\!/_{x_2} + R^{-1}(m' + d/2) \right] U G_F(\lambda(x_2; x_1)) \\ &= -\left[i\mathcal{D}\!\!\!/_{x_2} + m + R^{-1} \right] U G_F(\lambda(x_2; x_1)), \end{aligned}$$

which in the flat space limit $R \to \infty$ reduces to relationship (3.24) without R^{-1} term.

Extension to the covering space

In the expression (3.38) relating the spinor Feynman propagator to the scalar one, it can be observed that there are no non-integer powers of λ in the operator. By extension of the discussion in sections 2.5 it follows that operators such as $\partial^n_{y_M}$ do not change the phase factor as long as $n \in \mathbb{N}$. Therefore, the spinor propagator on the CadS covering space is simply given by the expression with the extended scalar propagator (3.22) at the right hand side:

$$\hat{\not{y}} S'^{(k)}_F(y_2; y_1) = \bar{\gamma}^B \left[R\partial_{y_2 B} + m' \hat{y}_{2\ B}\right] G^{(k)}_F(\lambda(y_2; y_1)). \tag{3.39}$$

The propagator only picks up a simple phase factor. Translation back to intrinsic fields and coordinates in the embedded CadS space is a straightforward exercise.

3.3 The CadS vector propagator

Many authors have studied spin-1 and higher fields in anti-de Sitter space. Fronsdal [Fro79] did so in his works on particles in a curved space. Gazeau [Gaz85, GH88] worked out massless, homogeneous propagator expressions and their group theoretical framework. Because $\mathrm{SO}(d-1,2)$ is not compact, it allows non-completely reducible representations, of which $D(2,1)$, representing massless spin-1 particles, is one (see section 2.2). When the curvature tends to zero, it can be shown that the contraction of this representation gives either the positive $D(0,+1)$ or negative $D(0,-1)$ helicity representations of the Poincaré group, but not both. Binegar, Fronsdal and Heidenreich [BFH83] constructed an adS massless vector theory that contracts to the Poincaré electromagnetic field by proposing two vector potentials that lose their dynamic independence in the contraction.

The present derivation of the vector propagator in d-dimensional anti-de Sitter space and its covering expands upon the treatment of the four-dimensional case by Janssen and Dullemond [JD87]. Although this section formed the core of the present author's masters thesis [dH92], it has since been significantly expanded and revised.

After a brief discussion of the vector propagator in Minkowski space-time, expressions will be found for the homogeneous two-point function. Compared to the scalar and spinor propagators, the construction turns out to be quite elaborate, starting longitudinal and multiple transversal solutions and combining them into a single result. Boundary conditions at infinity will guide the construction towards a unique result. The homogeneous expressions are then used to construct the inhomogeneous propagator. Finally, the latter is extended to the covering space.

Stueckelberg's Lagrangian for a massive vector field in $\mathbb{E}(d-1,1)$ reads [IZ80]:

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}\mu^2 A_\mu(x)A^\mu(x) - \frac{1}{2}c(\partial\cdot A(x))^2 \tag{3.40}$$

where μ is the mass, c the gauge fixing constant, and the field tensor $F = \mathrm{d}A$ the exterior derivative of the field potential: $F_{\mu\nu} = A_{[\nu,\mu]}$. The equations for the vector Feynman propagator D_F read ($x = x_2 - x_1$)

$$[\Box + \mu^2]D_{F\ \mu\nu}(x) - (1-c)\partial_\mu\partial^\rho D_{F\ \rho\nu}(x) = \eta_{\mu\nu}\delta^d(x). \tag{3.41}$$

In the $c = 1$ Feynman gauge, this reduces to a set of d independent scalar equations à la (3.1); the solution is $\eta_{\mu\nu}$ times the scalar propagator (3.3). In fact, the massless solution for arbitrary gauge is hardly more complex:

$$D^0_{F\ \mu\nu}(x)$$

$$= i\frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}\left[\frac{1+c}{c}\,\frac{1}{2}\,\frac{\eta_{\mu\nu}}{(-x^2+i\epsilon)^{\frac{d-2}{2}}}+\frac{1-c}{c}\,\frac{d-2}{2}\,\frac{x_\mu x_\nu}{(-x^2+i\epsilon)^{\frac{d}{2}}}\right].$$

This is the sum of a longitudinal $\sim p_\mu p_\nu = -\partial_\mu\partial_\nu$ and a transversal $p^\mu D_{F\ \mu\nu} = i\partial^\mu D_{F\ \mu\nu} = 0$, i.e. $\sim p^2\eta_{\mu\nu} - p_\mu p_\nu = -\eta_{\mu\nu}\Box + \partial_\mu\partial_\nu$ part:

$$\begin{aligned} &= i\frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}\,\frac{1}{2}\left\{\frac{1}{c}\left[\frac{\eta_{\mu\nu}}{(-x^2+i\epsilon)^{\frac{d-2}{2}}}+(d-2)\frac{x_\mu x_\nu}{(-x^2+i\epsilon)^{\frac{d}{2}}}\right]\right.\\ &\qquad\left.+\left[\frac{\eta_{\mu\nu}}{(-x^2+i\epsilon)^{\frac{d-2}{2}}}-(d-2)\frac{x_\mu x_\nu}{(-x^2+i\epsilon)^{\frac{d}{2}}}\right]\right\}\\ &= i\frac{\Gamma\left(\frac{d-4}{2}\right)}{16\pi^{\frac{d}{2}}}\left\{\frac{1}{c}\partial_\mu\partial_\nu-\left[-\eta_{\mu\nu}\Box+\partial_\mu\partial_\nu\right]\right\}\frac{1}{(-x^2+i\epsilon)^{\frac{d-4}{2}}}\end{aligned}$$

where only the longitudinal part is dependent on c — which makes sense, since it is precisely these longitudinal modes the gauge fixing procedure seeks to introduce. This structure shall be seen to recur in anti-de Sitter space propagator.

Because of the close parallels with what follows, it is worth exploring the massive homogeneous two-point function $iD_{\mu\nu}(x) = [A_\mu(x), A_\nu(0)]$ as well. The scalar two-point function G from (3.2) for $d = 4$ can be taken as a starting point[8]. It is possible to show that in analogy to the structure quoted above for the massless case,

$$D_{\mu\nu}(x) = \frac{1}{c}D^{\text{long}}_{\mu\nu}(x) + D^{\text{tr}}_{\mu\nu}(x) \tag{3.42}$$

where

$$\begin{aligned} D^{\text{tr}}_{\mu\nu}(x) &= \left[\eta_{\mu\nu}+\mu^{-2}\partial_\mu\partial_\nu\right]D(x,\mu)\\ D^{\text{long}}_{\mu\nu}(x) &= -(\mu/\sqrt{c})^{-2}\partial_\mu\partial_\nu D(x,\mu/\sqrt{c})\\ D(x,\mu) &\overset{\text{def}}{=} i\int\frac{\mathrm{d}^4k}{(2\pi)^3}e^{-ik\cdot x}\epsilon(k_4)\delta(k^2-\mu^2) = \operatorname{Im}G^{(\mu)}(s_{-\epsilon})\end{aligned}$$

Expanding G in a Neumann series ([AS84] 9.6.53), the function becomes[9]

$$D_{\mu\nu}(x) = \frac{1}{2\pi^2}\operatorname{Im}\left\{\eta_{\mu\nu}\left[-\frac{1}{s_{-\epsilon}}+\frac{1}{s_{-\epsilon}}\left(\frac{J_1(\sqrt{-\mu^2 s})}{\sqrt{-\mu^2 s}}-\frac{1}{c}\,\frac{J_1(\sqrt{-\mu^2 s/c})}{\sqrt{-\mu^2 s/c}}\right)\right.\right.$$

[8]It is possible to do the computation for arbitrary d, but the impossibility to do a Neumann series for general $D_\nu, \nu \notin \mathbb{N}$ makes the resulting expressions rather cumbersome, dragging the whole affair outside the scope of this section.

[9]The results presented here incorporate minor corrections on both Janssen's [JD87] and my own [dH92].

$$- \frac{\mu^2}{2} \frac{J_1(\sqrt{-\mu^2 s})}{\sqrt{-\mu^2 s}} \ln(\mu^2 s_{-\epsilon})$$
$$- \frac{1}{2s_{-\epsilon}} \Big(J_2(\sqrt{-\mu^2 s}) \ln(\mu^2 s_{-\epsilon}) - \frac{1}{c} J_2(\sqrt{-\mu^2 s/c}) \ln(\mu^2 s_{-\epsilon}/c) \Big) \Big]$$
$$+ x_\mu x_\nu \Big[\frac{2}{s_{-\epsilon}^2} \Big(\frac{J_1(\sqrt{-\mu^2 s})}{\sqrt{-\mu^2 s}} - \frac{1}{c} \frac{J_1(\sqrt{-\mu^2 s/c})}{\sqrt{-\mu^2 s/c}}$$
$$+ J_2(\sqrt{-\mu^2 s}) - \frac{1}{c} J_2(\sqrt{-\mu^2 s/c}) \Big)$$
$$+ \frac{1}{2s_{-\epsilon}} \Big(\mu^2 \frac{J_3(\sqrt{-\mu^2 s})}{\sqrt{-\mu^2 s}} \ln(\mu^2 s_{-\epsilon})$$
$$- \frac{1}{c} \mu^2/c \frac{J_3(\sqrt{-\mu^2 s/c})}{\sqrt{-\mu^2 s/c}} \ln(\mu^2 s_{-\epsilon}/c) \Big) \Big] \Big\} \tag{3.43}$$

The full Feynman propagator $D_{F\ \mu\nu}(x)$ is $1/2i$ times the above expression with the corrections for the Feynman path substituted, plus the (rather tedious) analytical terms from the Neumann series. Note that the most singular terms in $D^{\text{tr}}_{\mu\nu}(x)$ and $D^{\text{long}}_{\mu\nu}(x)$, which are of order $\mathcal{O}(s^{d/2})$, have exactly canceled in the full $D_{\mu\nu}(x)$. This cancellation of the transversal and longitudinal components' highest order singularity will recur in the anti-de Sitter case and will in fact be used to guide its construction.

The wave equation

The Lagrangian for a massive vector in curved space-time is [BD82]:

$$\mathcal{L}_{\mathcal{N}}(x) = [-g(x)]^{\frac{1}{2}} \Big\{ -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2} \mu^2 A_\mu(x) A^\mu(x) \Big\} \tag{3.44}$$

where $F^{\mu\nu}(x) \stackrel{\text{def}}{=} A^{[\nu;\mu]}(x) = A^{[\nu,\mu]}(x)$ is the electromagnetic field tensor. Using the action principle the wave equation

$$F^{\mu\nu}{}_{;\nu}(x) - \mu^2 A^\mu(x) = 0 \tag{3.45}$$

is obtained. If the mass μ is nonzero, the generalized Lorentz condition $A^\mu{}_{;\mu}(x) = 0$ holds.

In the case of anti-de Sitter space-time this can be considerably simplified by embedding in $\mathbb{E}(d-1,2)$ space[10]: the field potential A^μ becomes, using (2.16),

$$A^M(y(x)) = y^M{}_{,\mu} A^\mu(x) \tag{3.46}$$

[10] It is also possible to work the other way around, starting with the wave equations in $\mathbb{E}(d-1,2)$ and find the correspondence between the fields in $\mathbb{E}(d-1,2)$ and in CadS afterwards. This is the path followed by van Beveren et al. [vBRD86].

and the field tensor

$$\begin{aligned} F^{MN}(y(x)) &= y^M{}_{,\mu} y^N{}_{,\nu} F^{\mu\nu}(x) \\ &= P^M_K(x) P^N_L(x) (\mathcal{D}^K A^L(x) - \mathcal{D}^L A^K(x)) \\ &= (\mathcal{D}^M - y^{-2} y^M) A^N(x) - (\mathcal{D}^N - y^{-2} y^N) A^M(x) \end{aligned} \tag{3.47}$$

where $\mathcal{D}^M$ is the embedding of the covariant derivative. These expression can as usual be extended to a neighborhood of the hypersurface $\mathcal{N}$ in $\mathcal{M}$. In the remainder, both A^M and F^{MN} will be extended by defining them to be homogeneous with degree zero, i.e., they are to be a function of the projected coordinates $x(\hat{y})$ on $\mathcal{N}$ only, isotropic in $|y|$.

In the case of anti-de Sitter space, as has been demonstrated in section 2.4, the covariant derivative turns out to be just the tangential derivative. Given the y^2-independence of the extended field tensor $F^{MN}(y)$ and the potential $A^M(y)$, $\mathcal{D}^2$ effectively is proportional to the square of the generalized angular momentum operator M, defined in (2.34): $-y^2\mathcal{D}^2 \overset{\text{eff}}{=} M^2$.

The wave equation (3.45) then becomes

$$\begin{aligned} &\mathcal{D}_N F^{MN}(y) - \mu^2 A^M(y) = 0 \\ &\Leftrightarrow \quad \left[M^2 + (d-2)\right] A^M(y) - 2y^M \mathcal{D} \cdot A + y^2 \mathcal{D}^M \mathcal{D} \cdot A(y) \\ &\qquad = y^2 \mu^2 A^M(y) \end{aligned}$$

If $\mu \neq 0$, taking the divergence gives the generalized Lorentz condition $\mathcal{D} \cdot A(y) = 0$; suppressing the y dependence for clarity, the wave equation then reduces to

$$\left[M^2 - (y^2\mu^2 - d + 2)\right] A^M(y) = 0$$

which, in all its similarity to the scalar field equation (3.9), suggests a redefinition of the mass μ such as $y^2\mu^2 = (m+1)(m+d-2)$. Of course, there is a whole class of equivalent redefinitions on the physical space $\mathcal{N}$ differing only in their extension to $\mathcal{M}$; the choice

$$\mu^2 = R^{-2}(m+1)(m+d-2) \tag{3.48}$$

will turn out to be the most convenient. With this definition, the wave equation can be rewritten as

$$\begin{aligned} &\left[M^2 + (d-2)\right] A^M - 2y^M \mathcal{D} \cdot A + y^2 \mathcal{D}^M \mathcal{D} \cdot A \\ &\qquad = (m+1)(m+d-2) A^M \end{aligned} \tag{3.49}$$

and the Lorentz condition (for $\mu \neq 0$) reduces the wave equation to the equation for a set of $d+1$ independent scalar fields in (C)adS [DvB85, Fil89]:

$$\left[M^2 - m(m+d-1)\right] A^M = 0. \tag{3.50}$$

It can be seen that $m = \frac{2-d}{2} \overset{d\to 4}{=} -1$ gives the familiar massless case for scalar fields in anti-de Sitter space [DvB85]. This does, as may be expected, correspond to $\mu = 0$ (cf. 3.48). The corresponding Lagrangian is[11]

$$\mathcal{L}_{\mathcal{M}} = -\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}y^{-2}(m+1)(m+d-2)A_M A^M. \tag{3.51}$$

From this, it is but a small step to the generalization of Stueckelberg's Lagrangian:

$$\begin{aligned}\mathcal{L}_{\mathcal{M}} &= -\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\mu^2 A_M A^M - \frac{1}{2}c(\mathcal{D}\cdot A)^2 \\ &= -\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}y^{-2}(m+1)(m+d-2)A_M A^M \\ &\quad - \frac{1}{2}c(\mathcal{D}\cdot A)^2.\end{aligned} \tag{3.52}$$

The wave equation (3.49) changes into

$$\begin{aligned}&\left[M^2 - m(m+d-1)\right]A^M \\ &\quad -2y^M\mathcal{D}\cdot A + y^2(1-c)\mathcal{D}^M\mathcal{D}\cdot A = 0.\end{aligned} \tag{3.53}$$

Taking the divergence now results not in the Lorentz condition but in the field equation

$$\left[M^2 - (m+1)(m+d-2)/c\right]\mathcal{D}\cdot A = 0. \tag{3.54}$$

The homogeneous two point function

In anti-de Sitter space, the two point function should be a well behaved $SO(d-1,2)$ tensor. If it is defined to be the push-up to $\mathcal{M}$ (2.19) of the tensor $\kappa_{\mu\nu}(y;x)\mathrm{d}y^\mu \wedge \mathrm{d}x^\nu$ taking values in $T^*(y,\mathcal{N})\otimes T^*(x,\mathcal{N})$ (2.13), it must be in the kernel of $T(y,\mathcal{M}/\mathcal{N})\otimes T(x,\mathcal{M}/\mathcal{N})$. It must also be symmetric under the simultaneous interchange of $M \leftrightarrow N$ and $y \leftrightarrow x$. This leaves $d(d-3)/2$ independent components; in the $d \to 4$ limit, the tensor structure is given by the two basic tensors transverse in $\mathcal{M}$ with respect to y and x [Gaz85]:

$$\begin{aligned}&P_{ML}(y)P^L_{\ N}(x) \overset{\text{def}}{=} P_M(y)\cdot P_N(x) \\ &P_{ML}(y)x^L\, y^K P_{KN} \overset{\text{def}}{=} P_M(y)\cdot x\ y\cdot P_N(x)\end{aligned} \tag{3.55}$$

[11] It is possible to derive this Lagrangian directly from that in CadS space, as is done by Binegar et al. [BFH83]. However, this method brings some pitfalls – consider, for example, the domain of the integration in the action $I = \int \mathrm{d}^d x \mathcal{L}(x)$ when performed in CadS or $\mathbb{E}(d-1,2)$ – and it seems more straightforward to work through the wave equation instead.

Given the $SO(d-1,2)$ symmetry of (3.53), these tensor components are each proportional to a function of the two-point invariant $\lambda(y;x)$ defined in (2.36). Splitting up the κ_{MN} in this manner and taking $y_0 = y(0)$ (2.42) on the hyperboloid $y^2 = R^2$ (2.25) as a reference point, denoting $P_{0MN} \stackrel{\text{def}}{=} P_{MN}(y_0) = \eta_{MN} - \hat{y}_{0M}\hat{y}_{0N}$ (2.29):

$$\kappa_{MN}(z) = P_M \cdot P_{0N}\, f(z) + P_M \cdot \hat{y}_0\, \hat{y} \cdot P_{0N}\, g(z) \tag{3.56}$$

and from (3.53) the following set of coupled equations for $f(z)$ and $g(z)$ is obtained:

$$\begin{cases} \left[M^2 - (m+1)(m+d-2) + dc\right] f(z) - (1-c)zf'(z) \\ \quad + \left[-(d-1) + (d+1)c\right] zg(z) + (1-c)(1-z^2)g'(z) = 0 \\ \left[M^2 - (m+1)(m+d-2) + (d-1) + (d+1)c\right] g(z) \\ \quad + \left[-(d-1) + (d+3)c\right] zg'(z) + (1-c)(1-z^2)g''(z) \\ \quad + \left[-(d-1) + (d+1)c\right] f'(z) - (1-c)zf''(z) = 0. \end{cases} \tag{3.57}$$

The problem of solving these equations will be tackled in the next two sections by separating out the longitudinal and transversal solutions, then combining them while imposing physically reasonable constraints.

The longitudinal part

In the case of longitudinal functions satisfying (always on $y^2 = R^2$):

$$\begin{aligned} \kappa^{\text{long}}_{MN} &= R^2 \mathcal{D}_M \mathcal{D}_{0N} \kappa(z) \\ &= P_M \cdot P_{0N}\, \kappa'(z) + P_M \cdot \hat{y}_0\, \hat{y} \cdot P_{0N}\, \kappa''(z) \end{aligned} \tag{3.58}$$

the differential equations (3.57) become

$$\begin{cases} \left[M^2 - (m+1)(m+d-2) + dc\right] \kappa'(z) \\ \quad +(-n + (d+2)c)z\kappa''(z) + (1-c)(1-z^2)\kappa'''(z) = 0 \\ \left[M^2 - (m+1)(m+d-2) + (d-1) + 2(d+1)c\right] \kappa''(z) \\ \quad +(-n + (d+4)c)z\kappa''' + (1-c)(1-z^2)\kappa''''(z) = 0 \end{cases} \tag{3.59}$$

where the primes denote differentiation with respect to z. Using simple differentiation these can both be derived from the following equation:

$$\left[M^2 - (m+1)(m+d-2)/c\right] \kappa(z) = 0$$

or, recasting it in a form that resembles that of the scalar wave equation,

$$\left[M^2 - m'(m'+d-1)\right] \kappa(z) = 0, \tag{3.60}$$

where m' is related to m and the gauge fixing constant c as

$$m'(m'+d-1) \stackrel{\text{def}}{=} (m+1)(m+d-2)/c. \tag{3.61}$$

Note that the prime in m' serves just to distinguish it from m — it does not signify differentiation. The (four-dimensional) massless $m \to -1$ limit becomes the $m' \to 0$ limit. Note that m' exhibits the same $m' \leftrightarrow -m'-d+1$ symmetry as m.

Now, again substituting $\rho = z^2$ and working out M^2, this starts to look suspiciously like the all too familiar hypergeometrical equation:

$$\left[\rho(1-\rho)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + (\frac{1}{2} - \frac{d+1}{2}\rho)\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{m'(m'+d-1)}{4}\right]\kappa(\rho) = 0. \tag{3.62}$$

Solutions to this equation which are regular for $\lambda \to -\infty$ are:

$$\kappa_{m'<0} = (-\lambda)^{\frac{m'}{2}} F\left(-\tfrac{m'}{2}, -\tfrac{m'+d-2}{2}; -m' - \tfrac{d-3}{2}; \lambda^{-1}\right) \tag{3.63}$$

$$\kappa_{m'>-d+1} = (-\lambda)^{-\frac{m'+d-1}{2}} F\left(\tfrac{m'+d-1}{2}, \tfrac{m'+1}{2}; m' + \tfrac{d+1}{2}; \lambda^{-1}\right) \tag{3.64}$$

where $\lambda \stackrel{\text{def}}{=} 1 - z^2 \stackrel{\text{def}}{=} 1 - \rho$ (eqn. 2.37).

The two different solutions of the hypergeometrical equation correspond, of course, to the $m' \leftrightarrow -(m'+d-1)$ symmetry. In fact, attention could just as well have been limited to $m' > -d+1$, since simple substitution will yield the $m' < 0$ case. In subsequent sections just that will be done and the m' domain label will be suppressed from κ.

The solution found so far has good $\lambda \to \infty$ behavior, but a pole for $\lambda \to 0$; indeed, its massless limit is (see A.7)

$$\begin{aligned}\lim_{m'\to 0} \kappa &= \frac{d-1}{d-2}(-\lambda)^{-\frac{d-2}{2}} F\left(\tfrac{1}{2}, -\tfrac{d-2}{2}; -\tfrac{d-4}{2}; \lambda\right) + \pi^{-\frac{1}{2}}\Gamma\left(\tfrac{d+1}{2}\right)\Gamma\left(\tfrac{2-d}{2}\right) \\ &\stackrel{d\to 4}{=} \frac{3}{2}(-\lambda)^{-1} + \frac{3}{4}[-\ln(-\lambda) + 2\ln(2) - 1]\end{aligned}$$

Consequently, the longitudinal part has a $(-\lambda)^{-\frac{d}{2}}$ singularity. This is in complete analogy with the longitudinal part of (3.42) and indeed $\partial_\mu \partial_\nu D(x, \mu/\sqrt{c} \to 0)$ is recovered in the $R \to \infty$ limit of zero curvature. However, this is only the unphysical $\mathcal{O}(s^{d/2})$ term canceled in the full propagator $D_{\mu\nu}$. Because the whole solution is normalized by a factor $\sim \mu^{-2}$, naively taking the massless limit in intermediate results such as these is a recipe for disaster. It will be shown that the same is true for the anti-de Sitter propagator.

Finally, note that the other ($\kappa_{m'<0}$) solution becomes completely trivial in the $m' \to 0$ limit — a clear sign that the m' domains should be taken seriously.

The transversal part

Next consider transverse solutions which satisfy

$$\mathcal{D}^M \kappa^{\text{tr}}_{MN} = 0, \tag{3.65}$$

in this case (3.57) reduces to

$$\begin{cases} \left[M^2 - (m+1)(m+d-2) + d\right] f(z) + 2zg(z) = 0 \\ \left[M^2 - (m+1)(m+d-2) + 2d\right] g(z) \\ \qquad\qquad\qquad\qquad + 4zg'(z) + 2f'(z) = 0 \end{cases} \tag{3.66}$$

Substituting $\rho = z^2$ and writing out:

$$M^2 = -(1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + dz\frac{\mathrm{d}}{\mathrm{d}z} = -4\left[(1-\rho)\rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + (\frac{1}{2} - \frac{d+1}{2})\frac{\mathrm{d}}{\mathrm{d}\rho}\right]$$

yields the following set of equations:

$$\begin{cases} \left[\rho(1-\rho)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + (\frac{1}{2} - \frac{d+1}{2}\rho)\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{m(m+d-1)-2}{4}\right] f(\rho) \\ \quad - \frac{1}{2}zg(\rho) = 0 \\ \left[\rho(1-\rho)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + (\frac{1}{2} - \frac{d+1}{2}\rho)\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{m(m+d-1)-2}{4}\right] g(\rho) \\ \quad - zg'(\rho) - \frac{1}{2}f'(\rho) = 0 \end{cases} \tag{3.67}$$

Note that the primes still denote differentiation with respect to z! This is recognizable as a pair of coupled hypergeometrical equations; two independent solutions of these are given by[12]:

$$\begin{cases} f_1 = (m+d-1)^{-1}\left[F\left(-\frac{m-2}{2}, \frac{m+d-1}{2}; \frac{1}{2}; z^2\right) \right. \\ \qquad\qquad \left. + (m+d-2)F\left(-\frac{m}{2}, \frac{m+d-1}{2}; \frac{1}{2}; z^2\right)\right] \\ g_1 = z\left[(2-m)F\left(-\frac{m-4}{2}, \frac{m+d+1}{2}; \frac{3}{2}; z^2\right) \right. \\ \qquad\qquad \left. + (m+d-2)F\left(-\frac{m-2}{2}, \frac{m+d+1}{2}; \frac{3}{2}; z^2\right)\right] \end{cases} \tag{3.68}$$

and

$$\begin{cases} f_2 = z\left[(1-m)F\left(-\frac{m-3}{2}, \frac{m+d}{2}; \frac{3}{2}; z^2\right) \right. \\ \qquad\qquad \left. - m(m+d-2)F\left(-\frac{m-1}{2}, \frac{m+d}{2}; \frac{3}{2}; z^2\right)\right] \\ g_2 = \left[(1-m)F\left(-\frac{m-3}{2}, \frac{m+d}{2}; \frac{1}{2}; z^2\right) \right. \\ \qquad\qquad \left. + (m+d-2)F\left(-\frac{m-1}{2}, \frac{m+d}{2}; \frac{1}{2}; z^2\right)\right] \end{cases} \tag{3.69}$$

[12] This differs slightly from the corresponding results in [JD87], which are not entirely correct.

where ρ has been replaced again by z^2.

It is easy to verify that in the massless limit $m \to -1$ these solutions reduce to

$$\begin{cases} f_1^{m=-1} &= \frac{1}{\lambda^{\frac{d}{2}}} \\ g_1^{m=-1} &= \frac{dz}{\lambda^{\frac{d+2}{2}}} \end{cases} \tag{3.70}$$

The f_2 and g_2 reduce to linear combinations of Legendre functions and will not be quoted. For the flat space limit, these expressions are proportional to the $D^{\text{tr}}_{\mu\to 0}$ part of the Minkowski space propagator (3.42), but as in the longitudinal part, it is a mistake to take the massless limit like this — the terms in (3.70) will all cancel and the physically relevant part is actually proportional to μ^2.

The solutions found so far have a few pathological properties. Not only do they have higher order poles for $\lambda \to 0$; from the full expressions (3.68) and (3.69) it is also clear that they diverge for infinite spatial distances ($\lambda(y;y_0) \to \infty$ or, equivalently, $|x - x_0| \to \infty$). If the two-point function is to be physical, however, it should tend to zero for infinite separation between the points. But all is not lost, since there still is a whole family of solutions available to construct a linear superposition with satisfactory boundary conditions, which amount to demanding that the propagator approach zero sufficiently fast [AIS78, DvB85]. After finding a suitable combination of f functions, it should be established whether this combination also cures the same illnesses in g. These conditions will be invoked again later when combining the longitudinal and transversal parts into the final expression.

First thing to do is to analytically continue the f functions to extract their $\lambda \to -\infty$ behavior (cf. Abramowitz and Stegun [AS84] 15.3.8): f_1 (3.68) becomes

$$\begin{aligned} f_1 = \Gamma\left(\tfrac{3}{2}\right) \Bigg[& \frac{\lambda^{\frac{m-2}{2}} \Gamma\left(m + \frac{d-3}{2}\right)}{\Gamma\left(\frac{m+d+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)} \\ & \times \Bigg\{ \frac{m-1}{2} F\left(-\tfrac{m-2}{2}, -\tfrac{m+d-2}{2}; -m - \tfrac{d-5}{2}; \lambda^{-1}\right) \\ & \quad + \lambda(m+d-2)(m + \frac{d-3}{2}) \\ & \qquad \times F\left(-\tfrac{m}{2}, -\tfrac{m+d-2}{2}; -m - \tfrac{d-3}{2}; \lambda^{-1}\right) \Bigg\} \\ & + \frac{\lambda^{-\frac{m+d-1}{2}} \Gamma\left(-m - \frac{d-1}{2}\right)}{(m+d-1)\Gamma\left(-\frac{m+d-2}{2}\right)\Gamma\left(-\frac{m-2}{2}\right)} \end{aligned}$$

$$\times\Big\{2(-m-\frac{d-1}{2})F\Big(\tfrac{m-1}{2},\tfrac{m+d-1}{2};m+\tfrac{d-1}{2};\lambda^{-1}\Big)$$
$$-m(m+d-2)F\Big(\tfrac{m+d-1}{2},\tfrac{m+1}{2};m+\tfrac{d+1}{2};\lambda^{-1}\Big)\Big\}\Big]$$

The divergent part for the $m > -d+1$ case is

$$f^{\text{div}}_{1,m>-d+1} = \frac{\Gamma(\frac{3}{2})\Gamma(m+\frac{d-3}{2})}{\Gamma(\frac{m+d+1}{2})\Gamma(\frac{m-1}{2})}\lambda^{\frac{m-2}{2}}$$
$$\Big[F\Big(-\tfrac{m+d-2}{2},-\tfrac{m-2}{2};-m-\tfrac{d-5}{2};\lambda^{-1}\Big)$$
$$+\frac{2(m+d-2)(m+\frac{d-3}{2})}{m-1}\lambda F\Big(-\tfrac{m+d-2}{2},-\tfrac{m}{2};-m-\tfrac{d-3}{2};\lambda^{-1}\Big)\Big]$$

with a highest order term of $\lambda^{\frac{m-2}{2}}$. For f_2 one finds[13]

$$f_2 = -i\Gamma\left(\tfrac{3}{2}\right)\Bigg[\frac{\lambda^{\frac{m-2}{2}}\Gamma\left(m+\frac{d-3}{2}\right)}{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m}{2}\right)}$$
$$\times\Big\{-(m-1)F\Big(-\tfrac{m-2}{2},-\tfrac{m+d-2}{2};-m-\tfrac{d-5}{2};\lambda^{-1}\Big)$$
$$-2\lambda(m+d-2)(m+\frac{d-3}{2})$$
$$\times F\Big(-\tfrac{m}{2},-\tfrac{m+d-2}{2};-m-\tfrac{d-3}{2};\lambda^{-1}\Big)\Big\}$$
$$+\frac{\lambda^{-\frac{m+d-1}{2}}\Gamma\left(-m-\frac{d-1}{2}\right)}{\Gamma\left(-\frac{m+d-3}{2}\right)\Gamma\left(-\frac{m-1}{2}\right)}$$
$$\times\Big\{2(-m-\frac{d-1}{2})F\Big(\tfrac{m-1}{2},\tfrac{m+d-1}{2};m+\tfrac{d-1}{2};\lambda^{-1}\Big)$$
$$-m(m+d-2)F\Big(\tfrac{m+d-1}{2},\tfrac{m+1}{2};m+\tfrac{d+1}{2};\lambda^{-1}\Big)\Big\}\Bigg]$$

the $\lambda\to-\infty, m > -d+1$ divergent part of which is

$$f^{\text{div}}_{2,m>-d+1} = -i(1-m)\frac{\Gamma(\frac{3}{2})\Gamma(m+\frac{d-3}{2})}{\Gamma(\frac{m+d}{2})\Gamma(\frac{m}{2})}\lambda^{\frac{m-2}{2}}$$
$$\Big[F\Big(-\tfrac{m+d-2}{2},-\tfrac{m-2}{2};-m-\tfrac{d-5}{2};\lambda^{-1}\Big)$$
$$+\frac{2(m+d-2)(m+\frac{d-3}{2})}{m-1}\lambda F\Big(-\tfrac{m+d-2}{2},-\tfrac{m}{2};-m-\tfrac{d-3}{2};\lambda^{-1}\Big)\Big]$$

[13]using Abramowitz and Stegun 15.5.8, $F(a,b;c;\lambda^{-1}) = (\frac{\lambda}{-z^2})^{a+b-c}F(c-a,c-b;c;\lambda^{-1})$ which can be used to absorb the z into the hypergeometric functions F, which is where the factor i comes from.

Here too, of course, the $m < 0$ case can be safely disregarded because of the symmetry in m. The $m > -d + 1$ label will be omitted in the rest of the discussion.

A glance at the divergent parts f_1^{div} and f_2^{div} reveals that they will cancel nicely in the following linear combination of f_1 and f_2:

$$f \sim -2\Gamma\left(\tfrac{m+1}{2}\right)\Gamma\left(\tfrac{m+d+1}{2}\right)f_1 - i\Gamma\left(\tfrac{m}{2}\right)\Gamma\left(\tfrac{m+d}{2}\right)f_2. \tag{3.71}$$

and the following solution results[14]:

$$\begin{aligned} f = {} & (-\lambda)^{-\frac{m+d-1}{2}} \\ & \Big[m(m+d-2)F\Big(\tfrac{m+1}{2}, \tfrac{m+d-1}{2}; m + \tfrac{d+1}{2}; \lambda^{-1}\Big) \\ & \quad + (2m+d-1)\, F\Big(\tfrac{m-1}{2}, \tfrac{m+d-1}{2}; m + \tfrac{d-1}{2}; \lambda^{-1}\Big)\Big] \end{aligned} \tag{3.72}$$

It is evident that it is regular for $\lambda \to -\infty$ if $m > -d + 1$. Note that in the massless limit, f_1 retains only *convergent* terms while f_2 becomes fully *divergent*. Indeed, upon closer inspection one can see that upon taking the limit $m \to -1$ the linear combination becomes $f^{m=-1} = (d-2)(-)^{-\frac{d}{2}} f_1$.

Remains to be verified if the same linear combination rids the g component of its divergences at the same time. Analytically continuing g_1 and g_2 from (3.68) and (3.69) yields

$$\begin{aligned} g_1 = {} & -i\Gamma\left(\tfrac{3}{2}\right)\Bigg[\frac{\lambda^{\frac{m-3}{2}}\Gamma\left(m + \frac{d-3}{2}\right)}{\Gamma\left(\frac{m+d+1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)} \\ & \times\Big\{-(m-2)F\Big(-\tfrac{m-3}{2}, -\tfrac{m+d-1}{2}; -m - \tfrac{d-5}{2}; \lambda^{-1}\Big) \\ & \quad - 2\lambda\frac{(m+d-2)(m+\frac{d-3}{2})}{m-1} \\ & \qquad \times F\Big(-\tfrac{m-1}{2}, -\tfrac{m+d-1}{2}; -m - \tfrac{d-3}{2}; \lambda^{-1}\Big)\Big\} \\ & + \frac{\lambda^{-\frac{m+d}{2}}\Gamma\left(-m - \frac{d-1}{2}\right)}{\Gamma\left(-\frac{m+d-2}{2}\right)\Gamma\left(-\frac{m-2}{2}\right)} \\ & \times\Big\{2(-m - \frac{d-1}{2})F\Big(\tfrac{m-2}{2}, \tfrac{m+d}{2}; m + \tfrac{d-1}{2}; \lambda^{-1}\Big) \\ & \quad + (m+d-2)F\Big(\tfrac{m+d}{2}, \tfrac{m}{2}; m + \tfrac{d+1}{2}; \lambda^{-1}\Big)\Big\}\Bigg] \end{aligned}$$

[14] To be exact, for the sake of simplicity the whole expression was multiplied with the constant

$$-\frac{2(-)^{-\frac{m+d-1}{2}}}{\Gamma(\frac{1}{2})\Gamma(-m - \frac{d-1}{2})}\left[\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(-\frac{m-2}{2}\right)\Gamma\left(-\frac{m+d-2}{2}\right)} + \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(-\frac{m-1}{2}\right)\Gamma\left(-\frac{m+d-3}{2}\right)}\right]^{-1}$$

and

$$
\begin{aligned}
g_2 = \Gamma\left(\tfrac{1}{2}\right) & \left[\frac{\lambda^{\frac{m-3}{2}} \Gamma\left(m + \frac{d-3}{2}\right)}{\Gamma\left(\frac{m+d}{2}\right) \Gamma\left(\frac{m-2}{2}\right)} \right. \\
& \times \left\{ (m-1) F\left(-\tfrac{m-3}{2}, -\tfrac{m+d-1}{2}; -m - \tfrac{d-5}{2}; \lambda^{-1}\right) \right. \\
& \quad - 2\lambda \frac{(m+d-2)(m+\frac{d-3}{2})}{m-2} \\
& \quad\quad \left. \times F\left(-\tfrac{m-1}{2}, -\tfrac{m+d-1}{2}; -m - \tfrac{d-3}{2}; \lambda^{-1}\right) \right\} \\
& + \frac{\lambda^{-\frac{m+d}{2}} \Gamma\left(-m - \frac{d-1}{2}\right)}{\Gamma\left(-\frac{m+d-1}{2}\right) \Gamma\left(-\frac{m-1}{2}\right)} \\
& \times \left\{ 2(-m - \frac{d-1}{2}) F\left(\tfrac{m-2}{2}, \tfrac{m+d}{2}; m + \tfrac{d-1}{2}; \lambda^{-1}\right) \right. \\
& \quad \left. \left. + (m+d-2) F\left(\tfrac{m+d}{2}, \tfrac{m}{2}; m + \tfrac{d+1}{2}; \lambda^{-1}\right) \right\} \right]
\end{aligned}
$$

Apparently the linear combination of (3.71) here, too, results in cancellation of the divergent parts, i.e. all powers of λ greater than zero:

$$
\begin{aligned}
g = {} & (m+d-1)(-\lambda)^{-\frac{m+d}{2}} \\
& \left[(m+d-2) F\left(\tfrac{m}{2}, \tfrac{m+d}{2}; m + \tfrac{d+1}{2}; \lambda^{-1}\right) \right. \\
& \quad \left. - (2m+d-1)\, F\left(\tfrac{m-2}{2}, \tfrac{m+d}{2}; m + \tfrac{d-1}{2}; \lambda^{-1}\right) \right]
\end{aligned}
\tag{3.73}
$$

Therefore the transversal solution constructed so far is free of unphysical $\lambda \to -\infty$ divergences. Of course this means one can make the same observations for the massless limit of the g solutions as those made in the case of the f solutions.

The complete solution

Two solutions for the homogeneous vector equation have been derived so far: one longitudinal (3.64), the other transversal (3.72 and 3.73). Both, however, retain a pole of order $\lambda^{-\frac{d-2}{2}} \approx (\Delta x)^{2-d}$ when the two reference points approach each other. From a physical point of view one imposes boundary conditions such that the degree of divergence is minimal.

Encouraged by the successes so far, a line of attack similar to that in the previous chapters will be pursued. After analytically continuing κ, f and g into the $\lambda \approx 0$ domain, one looks for a linear combination of longitudinal and transversal solutions taming the $\lambda \to 0$ singularity. The calculational detail has been relegated to Appendix A.2 for brevity.

Looking at the expansion around $\lambda \to 0$, the most divergent terms for the $P_M \cdot P_{0N}$ part are (cf. 3.56)

$$\kappa'^{\text{div}} = -\frac{\Gamma\left(m' + \frac{d+1}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)} \frac{d-2}{(-\lambda)^{\frac{d}{2}}} + \ldots \tag{3.74}$$

$$f^{\text{div}} = \frac{\Gamma\left(m + \frac{d+1}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)} \frac{d-2}{(-\lambda)^{\frac{d}{2}}} + \ldots \tag{3.75}$$

therefore the linear combination

$$\begin{aligned} \mathcal{F}(\lambda) &= \frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(m + \frac{d+1}{2}\right)} f(\lambda) + \frac{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}{\Gamma\left(m' + \frac{d+1}{2}\right)} \kappa'(\lambda) \\ &= \frac{1}{2}\Gamma\left(\frac{d-2}{2}\right) \frac{c+1}{c} \frac{(m+1)(m+d-2)}{(-\lambda)^{\frac{d-2}{2}}} + \ldots \end{aligned} \tag{3.76}$$

forms (up to the normalization) a solution where this highest order singularity has been canceled, leaving (for $d \to 4$) only a simple pole in λ in the origin.

$\mathcal{F}$ has some further properties worth noting. First, the pole appears to vanish for the massless limit $m \to -1$. Upon closer examination it turns out that all terms of $\mathcal{F}$ (see (A.12) for the terms up to order $\mathcal{O}(\lambda^0)$) are proportional to the mass squared $(m+1)(m+d-2) = m'(m'+d-1)c = \mu^2 R^2$; indeed, this factor will eventually be absorbed in the propagator's normalization. Apparently the pole does not vanish in the massless limit after all. Furthermore the singularity vanishes when the gauge choice $c = -1$ is made. On the other hand, it blows up completely for the $c = 0$ gauge, underlining the important role played by the gauge fixing term.

Of course, not much was gained unless the very same linear combination tames the singularity in the $y^{-2} y \cdot P_{0N} y_0 \cdot P_M$ part of the two-point function as well:

$$\begin{aligned} \mathcal{G}(\lambda) &= \frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(m + \frac{d+1}{2}\right)} g(\lambda) + \frac{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}{\Gamma\left(m' + \frac{d+1}{2}\right)} \kappa''(\lambda) \\ &= \Gamma\left(\frac{d}{2}\right) \frac{c-1}{c} \frac{(m+1)(m+d-2)}{(-\lambda)^{\frac{d}{2}}} + \ldots \end{aligned} \tag{3.77}$$

One observes that the highest order singularity has indeed been canceled; like $\mathcal{F}$, the expression is found (see (A.13) for further terms and full expressions) to be proportional to $\mu^2 R^2$ – as, indeed, $\mathcal{G}$ in its entirety – which will be seen to normalize out. The gauge choice $c = 0$ is singular here as well.

The gauge choice removing the highest order singularity in $\mathcal{G}$ this time is the Feynman gauge $c = 1$. All of this reveals close parallels with the well-known Minkowski space results.

For the $d = 4$ massless vector propagator, a remarkable simplification is achieved in the gauge $c = 1/3$: the logarithmic terms which crop up in the $d \to 4$ limit disappear. This is a well-known phenomenon described in detail by Gazeau [Gaz85] indicating a discontinuity in the Inönü-Wigner contraction. As the loss of conformal invariance for $d \neq 4$ gives "anomalous" mass terms in the propagator (A.14) even in the massless case, this phenomenon is particular to $d = 4$.

The inhomogeneous two point function

Following the analysis of the solution to the homogeneous equation (3.53) it is now time to turn the attention to the full Feynman propagator $iR^2 K_F^{MN}(y_2, y_1) = \langle 0|TB^M(y_2)B^{\dagger N}(y_1)|0\rangle$ for vector fields, satisfying the inhomogeneous equation

$$\begin{aligned}&\left[M_{y_2}^2 - m(m+d-1)\right] K_F^{MN}(y_2(x_2), y_1(x_1))\\ &\quad -2y_2^M \mathcal{D}_{y_2}^L K_{FL}{}^N(y_2(x_2), y_1(x_1))\\ &\quad +y_2^2(1-c)\mathcal{D}_{y_2}^M \mathcal{D}_{y_2}^L K_{FL}{}^N(y_2(x_2), y_1(x_1))\\ &\qquad = -R^2 P^{MN}(y_2(x_2))\delta^d(x_2 - x_1). \qquad (3.78)\end{aligned}$$

The procedure is analogous to that followed for the scalar propagator in section 3.1. The above expression for the Feynman propagator implies, $x, y \in \mathcal{M}$,

$$\int_{\Sigma_{\pm a}} \mathrm{d}^{d-1}\vec{y}\ \frac{\partial}{\partial y^4} K_F^{MN}(y;x)\bigg|_{y^4=\pm a} = \epsilon(a) P^{MN}(x). \qquad (3.79)$$

The following paragraphs will work towards this result. To start with, an infinitesimal time translation $y \to e^{\pm\frac{\epsilon}{2}M_{54}} y$ and therefore $\lambda \to \lambda_{\pm\epsilon}$ (2.46-2.48) serves to define the integration paths in which the two-point function plays a role. Evaluating the integration in (3.79) for $K^{MN}(y_{\pm\epsilon}; x)$, it will be shown that only the pole terms in κ (cf. 3.56, 3.76, 3.77)

$$\begin{aligned}\kappa^{MN}(y;x) &= \frac{1}{2}\Gamma\left(\tfrac{d-2}{2}\right)\mu^2 R^2 P^M(y)\cdot P^N(x)\frac{c+1}{c}(-\lambda)^{-\frac{d-2}{2}}\\ &\quad + \Gamma\left(\tfrac{d}{2}\right)\mu^2 R^2 P^M(y)\cdot x\ y\cdot P^N(x)\frac{c-1}{c}(-\lambda)^{-\frac{d}{2}} + \mathcal{O}(\lambda^{-\frac{d-4}{2}})\\ &= \frac{1}{2}\Gamma\left(\tfrac{d-2}{2}\right)\mu^2 R^2 P^{NM}(x)\frac{c+1}{c}(-\lambda)^{-\frac{d-2}{2}}\\ &\quad - \Gamma\left(\tfrac{d}{2}\right)\mu^2 R^2\ \hat{\varepsilon}^M \hat{\varepsilon}^N \frac{c-1}{c}(-\lambda)^{-\frac{d}{2}} + \mathcal{O}(\hat{\epsilon}^{3-d}) \qquad (3.80)\end{aligned}$$

do contribute (remember $\hat{\varepsilon} \stackrel{\text{def}}{=} \hat{y} - \hat{x}$). For the first term, this is evident for the same reason the higher order terms in the scalar integration (3.16) did not contribute in the $\epsilon \to 0$ limit. In fact, the integral is identical to that for the scalar function, leading in the $c = 1$ Feynman gauge directly to the result

$$\lim_{\epsilon\to 0} \frac{\partial}{\partial y^4} \kappa^{MN}(y_{\pm\epsilon}; x)\bigg|_{y^4=0} = \pm 2\pi^{\frac{d}{2}} i\mu^2 R^d P^{MN}(x)\delta^{d-1}(\vec{y} - \vec{x}). \qquad (3.81)$$

This expression is actually valid for every gauge choice. To see this, the second term's contribution needs to be evaluated, which takes a bit more work. Only even terms survive the integration, so $\hat{\varepsilon}^M \hat{\varepsilon}^N \stackrel{\text{eff}}{=} P^M{}_P(x) P^N{}_Q(x) \hat{\varepsilon}^P \hat{\varepsilon}^Q + \frac{1}{4}\hat{x}^M \hat{x}^N (\hat{\varepsilon}^2)^2$ (see B.5). Manifest $SO(d-1,2)$ invariance means that the integration can be evaluated for x equal to the reference point $y_0 = (\vec{0}, 0, R)$ without loss of generality (2.41, 2.42). In that case, for $M, N \in 1 \dots d-1$, the first of these terms vanishes and the second does not contribute in the $\epsilon \to 0$ limit. For $M = N \in 1 \dots d-1$, the contribution comes from the $\hat{\varepsilon}^P \hat{\varepsilon}^Q$ term only and is proportional to

$$\begin{aligned}
&\int_{\Sigma_0'} \mathrm{d}^{d-1}\vec{y}\,(\hat{\varepsilon}^{P=Q})^2 \,\frac{\partial}{\partial y^4}(-\lambda_{\pm\epsilon}(y; y_0))^{-\frac{d}{2}}\bigg|_{y^4=0} \\
&= -\frac{1}{d-1}\int_{\Sigma_0'} \mathrm{d}^{d-1}\vec{y}\frac{r^2}{R^2}\,\frac{\partial}{\partial y^4}(-\lambda_{\pm\epsilon}(y; y_0))^{-\frac{d}{2}}\bigg|_{y^4=0} \\
&= \mp i\epsilon\frac{d}{2R^3(d-1)}\int \mathrm{d}^{d-2}\Omega \int \mathrm{d}r \frac{r^d\sqrt{1 + r^2/R^2}}{(r^2/R^2(1+\epsilon^2/4) + \epsilon^2/4)^{\frac{d}{2}}} \\
&= \mp i\frac{dR^{d-2}}{2(d-1)}\Omega_{d-1} B(\frac{d+1}{2}, \frac{1}{2}) + \mathcal{O}(\epsilon).
\end{aligned}$$

The only other non-vanishing contribution is for $M = N = d$. Realizing that the contraction of the integral with $P_{MN}(y_0)$ yields, using (2.38), precisely $d-2$ times the abovementioned result,

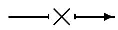
$$\begin{aligned}
&P_{MN}(y_0)\int_{\Sigma_0'} \mathrm{d}^{d-1}\vec{y}\,\frac{\partial}{\partial y^4}\hat{\varepsilon}^M \hat{\varepsilon}^N (-\lambda_{\pm\epsilon}(y; y_0))^{-\frac{d}{2}}\bigg|_{y^4=0} \\
&= -\int_{\Sigma_0'} \mathrm{d}^{d-1}\vec{y}\,\frac{\partial}{\partial y^4}(-\lambda_{\pm\epsilon}(y; y_0))^{-\frac{d-2}{2}}\bigg|_{y^4=0} + \mathcal{O}(\epsilon) \\
&= \mp 2\pi^{\frac{d}{2}} i \frac{R^{d-2}}{\Gamma\left(\frac{d-2}{2}\right)} + \mathcal{O}(\epsilon)
\end{aligned}$$

it is clear that the $M = N = d$ case must give a contribution equal to the $d-1$ others, so all in all the integral is simply proportional to $P^{MN}(y_0)$:

$$\int_{\Sigma_0'} \mathrm{d}^{d-1}\vec{y}\ \frac{\partial}{\partial y^4}\hat{\varepsilon}^M\hat{\varepsilon}^N(-\lambda_{\pm\epsilon}(y;y_0))^{-\frac{d}{2}}\bigg|_{y^4=0}$$
$$= \mp\frac{P^{MN}(y_0)}{d-2}2\pi^{\frac{d}{2}}i\frac{R^{d-2}}{\Gamma\left(\frac{d-2}{2}\right)} + \mathcal{O}(\epsilon).$$

Comparison with (3.80) shows that, apart from its c dependency, the second term's contribution to the integral is identical to the first. Equation (3.81), then, is proportional to $\frac{c+1}{c} + \frac{c-1}{c} = 2$ and therefore independent of the gauge fixing parameter c.

Given the close parallels of this result with that for the scalar propagator, it will not surprise that the remainder of the procedure is completely analogous. Defining

$$K^{MN}(y;x) \stackrel{\text{def}}{=} \frac{1}{2\pi^{\frac{d}{2}}\mu^2R^d}\mathrm{Re}\,\kappa^{MN}(y_{+\epsilon};x) =$$ 

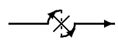
$$\bar{K}^{MN}(y;x) \stackrel{\text{def}}{=} \frac{1}{2\pi^{\frac{d}{2}}\mu^2R^d}\mathrm{Im}\,\kappa^{MN}(y_{+\epsilon};x) = -\frac{1}{2i}$$  (3.82)

the retarded and advanced propagators can be constructed as

$$K^{MN}_{\text{ret,adv}}(y;x) = \pm\theta(\pm(t_y - t_x))\bar{K}^{MN}(y;x) = \begin{cases} \frac{1}{i} \\ \frac{1}{i} \end{cases}$$   . (3.83)

Finally, the Feynman vector propagator is

$$\begin{aligned} K^{MN}_F(y;x) &= -\frac{1}{2i}\left[K^{MN}(y;x) - i\,\mathrm{sgn}(t_y - t_x)\bar{K}^{MN}(y;x)\right] \\ &= -\frac{1}{2i}\,\frac{1}{2\pi^{\frac{d}{2}}\mu^2R^d}\kappa^{MN}(y_F;x) \end{aligned} \tag{3.84}$$

where y_F denotes a modification of the integration path to  or, equivalently,  . These expressions are all valid for any choice of the gauge fixing parameter except $c = 0$.

Finally, it is worth examining the $d \to 4$ limit. Examination of the higher order terms in (A.14) shows that the $1/(d-4)$ poles cancel in the $d \to 4$ limit and, as shown for the scalar propagator (3.12) and the longitudinal κ, give rise to $\ln(-\lambda)$ terms, bringing the vector propagator (3.56, 3.76, 3.77) in Hadamard form:

$$K^{(0)\,MN}_{F,d=4}(y;x) = -\frac{1}{8i\pi^2R^2}\bigg\{P^M(y)\cdot P^N(x)\bigg[\frac{1+c}{c}(-\lambda_F)^{-1}$$

$$
\begin{aligned}
&+ a(m,c,\lambda)\ln(-\lambda/4) + b(m,c,\lambda)\Big] \\
&+P^M(y)\cdot\hat{x}\;\hat{y}\cdot P^N(x)\Big[-2\frac{1-c}{c}(-\lambda_F)^{-2} \\
&-\frac{1}{4}\Big\{-\frac{1-c^2}{c^2}(m+1)(m+2)+2\frac{1+c}{c}\Big\}(-\lambda_F)^{-1} \\
&+ d(m,c,\lambda)\ln(-\lambda/4) + e(m,c,\lambda)\Big]\Big\}
\end{aligned}
$$

where a, b, d and e are analytical functions of λ. In the massless limit $a(m,c,\lambda)$ and $d(m,c,\lambda)$ tend to zero for the special gauge choice $c = 1/3$, which corresponds to $c = 2/3$ in [BFH83, Gaz85].

Extension to the covering space

Up to now, the computation has taken place in plain anti-de Sitter space without concern for its universal covering CadS. Section 2.5 showed that analytical continuation is very straightforward. The only differences in the two-point function on different sheets is a simple phase factor that can be determined from the behavior at large λ (2.49). The integration path for the Feynman propagator is extended as defined in figure 3.1.

From (3.64) it is clear that the generating function for the longitudinal part of the solution extends to

$$\kappa^{(k)}(\lambda_{\pm\epsilon}(y;x)) = e^{\mp\pi i(m'+d-1)(k_y-k_x)}\kappa^{(0)}(\lambda_{\pm\epsilon}(y;x))$$

so the two derivatives with respect to z, κ' and κ'', will be

$$
\begin{aligned}
\kappa'^{(k)}(\lambda_{\pm\epsilon}(y;x)) &= 2\sqrt{1-\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\kappa^{(k)}(\lambda_{\pm\epsilon}(y;x)) \\
&= e^{\mp\pi i(m'+d)(k_y-k_x)}\kappa'^{(0)}(\lambda_{\pm\epsilon}(y;x)) \\
\kappa''^{(k)}(\lambda_{\pm\epsilon}(y;x)) &= \Big[4(1-\lambda)\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}+2\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big]\kappa^{(k)}(\lambda_{\pm\epsilon}(y;x)) \\
&= e^{\mp\pi i(m'+d-1)(k_y-k_x)}\kappa''^{(0)}(\lambda_{\pm\epsilon}(y;x))
\end{aligned}
$$

From (3.72) and (3.73) the transversal solution becomes

$$
\begin{aligned}
f^{(k)}(\lambda_{\pm\epsilon}(y;x)) &= e^{\mp\pi i(m+d-1)(k_y-k_x)}f^{(0)}(\lambda_{\pm\epsilon}(y;x)) \\
g^{(k)}(\lambda_{\pm\epsilon}(y;x)) &= e^{\mp\pi i(m+d)(k_y-k_x)}g^{(0)}(\lambda_{\pm\epsilon}(y;x))
\end{aligned}
$$

The analytic continuation to the covering space of the full linear combination of (3.76) and (3.77) canceling out the highest order singularities in these solutions, then, is rather more complex than encountered so far, as the phase factors for the two terms (κ' and f, and κ'' and g, respectively) are generally different for arbitrary m and c. As a consequence, the homogeneous solution $\kappa^{(k)\ MN}(\lambda_{\pm\epsilon})$ has "untamed" singularities on sheets $k_y \neq k_x$. The same problem affects the Feynman propagator:

$$
\begin{aligned}
K_F^{(k)\ MN}(y;x) &= -\frac{1}{2i}\,\frac{1}{2\pi^{\frac{d}{2}}\mu^2 R^d} \\
&\times\Bigg\{P^M(y)\cdot P^N(x)\Big[e^{-\pi i(m+d-1)|k_y-k_x|}\frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(m+\frac{d+1}{2}\right)}f^{(0)}(\lambda_F) \\
&\quad + e^{-\pi i(m'+d)|k_y-k_x|}\frac{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}{\Gamma\left(m'+\frac{d+1}{2}\right)}\kappa'^{(0)}(\lambda_F)\Big] \\
&+P^M(y)\cdot\hat{x}\ \hat{y}\cdot P^N(x)\Big[e^{-\pi i(m+d)|k_y-k_x|}\frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(m+\frac{d+1}{2}\right)}g^{(0)}(\lambda_F) \\
&\quad + e^{-\pi i(m'+d-1)|k_y-k_x|}\frac{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}{\Gamma\left(m'+\frac{d+1}{2}\right)}\kappa''^{(0)}(\lambda_F)\Big]\Bigg\} \qquad (3.85)
\end{aligned}
$$

This does, however, not give rise to physical anomalies as only the $k_y = k_x$ sheet really contributes to physical processes, contributions from the infinities on other sheets being strongly suppressed (see section B.2).

chapter 4

Processes

Quantum field theory in curved space-time has been the subject of much and varied theoretical research; it is impossible to do the subject justice here. Much of it is now textbook material [BD82, Ful89]. However, to the best of the author's knowledge, no-one has yet subjected anti-de Sitter QED to the standard regularization and renormalization programme. Filthaut and Dullemond [FD91] showed the renormalizability of ϕ^4 theory in anti-de Sitter space. Given the remarkable similarities with physics in Minkowski space, it is likely that the entirety of conventional quantum field theory can be extended to anti-de Sitter space. Far from being an academic exercise, this would give insight in the influence of strong background curvature on fundamental physical processes, conditions which may have existed at some stages in the early universe. This chapter is part of an attempt to extend the results of Filthaut and Dullemond to quantum electrodynamics. Another important inspiration for this programme is the work of Drummond and Shore [DS79] in spherical space-time.

Unitarity in general curved space-times is not self-evident. For a large class of space-times, however, one can generalize the LSZ reduction and configuration space Cutkosky rules in a relatively straightforward manner, as Friedman et al showed in a programme for ϕ^4 theory [FPS92b]. Provided the theory is also renormalizable, these ingredients can be used to prove unitarity. The subject of field quantization in anti-de Sitter space has been explored by many authors [Fro74, FH75, AIS78, GB94] and will not be repeated here. QED can be formulated in much the same way as in flat space, with the obvious substitution of covariant derivatives for normal derivatives. Much of the flat-space theoretical framework also carries over directly. The theory can be formulated using the path integral formulation. Interactions between the electrodynamic field and the electron can be introduced using minimal coupling and give a theory with local gauge invariance. The Feynman rules are therefore essentially identical, as is the procedure of regularization and renormalization. Some of the simpler Ward identities can be easily proven to be as valid in anti-de Sitter space as it is in flat space, with the derivative again becoming a covariant one.

Due to limitations in the methods employed, only the $1/(4-d)$ infinities will be calculated. These can then be used to renormalize the theory to the first order in a minimal subtraction scheme. The diagrams in this chapter will be evaluated using configuration space methods and regularized using dimensional regularization [tHV72, Dru75]. It may be useful to recall at this point why the momentum representation that is so convenient in traditional Minkowski quantum field theory becomes so much more cumbersome in curved space-times. Consider that perennial favorite among toy models, ϕ^3

theory, in anti-de Sitter space. The free field satisfies the wave equation (3.5)

$$\left[(i\mathcal{D})^2 - m_0^2\right]\phi(x) = 0$$

Call the set of normalized $(i\mathcal{D})^2 = R^{-2}M^2$ eigenfunctions[1] $\phi_{\sigma,K}(y)$. From (2.10), defining $\sigma = E_0 - d + 1$, their eigenvalues are $\sigma(\sigma + d - 1)$. K is actually a composite index collecting all other state information. A $g\phi^3(y)/3!$ interaction term is then introduced to the Lagrangian (3.4) and its physical impact evaluated using the usual perturbative methods. The configuration space Feynman rules for this field are:

1. To each vertex, attach a factor $-ig \int\limits_{\mathcal{N}} \mathrm{d}^d x[-g(x)]^{\frac{1}{2}}$.
2. For each line between internal vertices, there is a corresponding propagator $G_F(\lambda(x;y))$ (3.22).
3. For each external incoming particle, a factor $\phi_{\sigma,K}(x)$.
4. For each external outgoing particle, a factor $\phi_{-\sigma-d+1,\bar{K}}(x)$.
5. A factor N^{-1}, where N is the order of the diagram's symmetry group.

Take, for example, the one-loop bubble diagram in figure 4.1. The Feynman

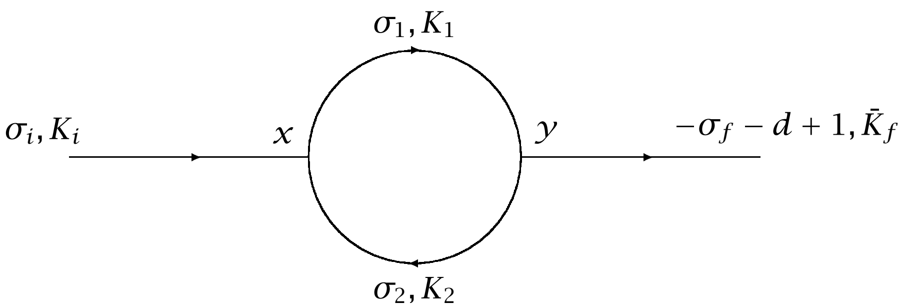


figure 4.1: The ϕ^3 bubble diagram

rules give the following expression for the truncated diagram:

$$(-ig)^2 \int\limits_{\mathcal{N}} \mathrm{d}^d x[-g(x)]^{\frac{1}{2}} \int\limits_{\mathcal{N}} \mathrm{d}^d y[-g(y)]^{\frac{1}{2}}$$
$$\times\phi_{\sigma_i,K_i}(x)G_F(x;y)G_F(y;x)\phi_{-\sigma_f-d+1,\bar{K}_f}(y) \tag{4.1}$$

[1]The actual expressions for these eigenfunctions are irrelevant in the present discussion.

The momentum-space approach may be generalized to adS by applying the transform $f(\sigma,K) = \int_{\mathcal{N}} \mathrm{d}^d x f(x)\phi_{\sigma,K}(x)$. Equation (3.10) for the scalar propagator becomes algebraic and can be inverted[2]:

$$\begin{aligned} G_F(x;y) &= \sum_K \int \mathrm{d}\sigma\rho(\sigma)\phi_{\sigma,K}(x)G_F(\sigma)\phi_{-\sigma-d+1,\bar{K}}(y) \\ G_F(\sigma) &= \left(\sigma_\varepsilon(\sigma_\varepsilon + d - 1) - m_0^2\right)^{-1} \end{aligned}$$

However, this does not significantly reduce the complexity of the integral (4.1):

$$(-ig)^2 \sum_{K_1,K_2} \int \mathrm{d}\sigma_1\rho(\sigma_1)\mathrm{d}\sigma_2\rho(\sigma_2)G_F(\sigma_1)G_F(\sigma_2)f^x_{\sigma_1K_1\sigma_2K_2}f^y_{\sigma_1K_1\sigma_2K_2} \quad (4.2)$$

where the functions f

$$\begin{aligned} f^x_{\sigma_1K_1\sigma_2K_2} &= \int_{\mathcal{N}} \mathrm{d}^d x[-g(x)]^{\frac{1}{2}}\phi_{\sigma_i,K_i}(x)\phi_{\sigma_1,K_1}(x)\phi_{-\sigma_2-d+1,\bar{K}_2}(x) \\ f^y_{\sigma_1K_1\sigma_2K_2} &= \int_{\mathcal{N}} \mathrm{d}^d y[-g(y)]^{\frac{1}{2}}\phi_{-\sigma_f-d+1,\bar{K}_f}(y)\phi_{-\sigma_1-d+1,\bar{K}_1}(y)\phi_{\sigma_2,K_2}(y) \end{aligned}$$

are vertex factors. Compare this with Minkowski space, where the $\phi_{\sigma,K}(y)$ eigenfunctions become the plane waves e^{ikx} and, for example, the factor for vertex x would reduce to simple momentum conservation:

$$f^x_{k_1,k_2} = \int \frac{\mathrm{d}^4 x}{(2\pi)^4} e^{ik_ix}e^{ik_1x}e^{-ik_2x} = \delta^4(k_i + k_1 - k_2).$$

This happens independently of the type of interaction. In curved space, on the other hand, the expression for a given diagram includes a product of vertex factors $\prod f_{\{\sigma\}\{K\}}$ that depend on the type of interaction under consideration and that do not generally reduce to delta functions. They are the generalization of momentum conservation.

Therefore, the Feynman rules for ϕ^3 theory in transform space, are

1. To each vertex, attach a factor $-igf_{\{\sigma\}\{K\}}$.
2. For each internal line, $G_F(\lambda(\sigma))$.
3. An integration $\sum_K \int \mathrm{d}\sigma\rho(\sigma)$ over each internal loop transform variable.

[2] $\rho(\sigma)$ is a weight function following from the delta-function identity for the $\phi_{\sigma,K}(x)$ — see, for example, [Dav96].

4. A factor N^{-1}, where N is the order of the diagram's symmetry group.

Even for a simple maximally symmetric space such as anti-de Sitter space, the vertex factors are not trivial to evaluate. In the context of the present discussion, configuration space methods have the advantage of relative simplicity.

In multi-loop diagrams, it is not always trivial to compare the momentum-space renormalization procedure to the configuration-space one, as (sub-) graphs regular in momentum space do not have to be so in configuration space and vice versa [Col84]. In the simple diagrams considered in the next few sections, however, no such difficulties arise and the results can, in the limit for zero curvature, be compared directly to the textbook results for momentum space.

Although they are essentially identical to the flat space rules, it is useful to quote the configuration space Feynman rules for adS QED here:

1. To each vertex, attach a factor $-ie\gamma^\mu \int\limits_{\mathcal{N}} \mathrm{d}^d x[-g(x)]^{\frac{1}{2}}$.
2. For each electron line between internal vertices there is a propagator $S_F(\lambda(x;y))$ (3.39).
3. For each photon line between internal vertices there is a propagator $K_{F\ \mu\nu}(x;y)$ (3.85)
4. A factor -1 for each closed fermion loop.
5. An overall sign depending on the configuration of the external fermion lines compared to the arguments of the propagators.
6. A factor N^{-1}, where N is the order of the diagram's symmetry group.

4.1 Vacuum polarization

Consider a single vacuum polarization bubble as in figure 4.2. From the

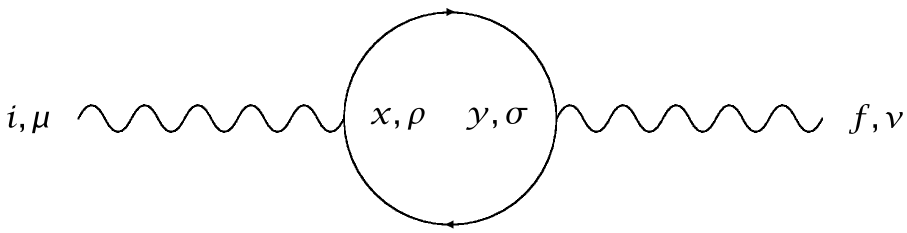


figure 4.2: The vacuum polarization bubble.

Feynman rules quoted in the introduction, this gives a contribution to the photon propagator of:

$$\begin{aligned} K^{[1]}_{\nu\mu}(f;i) &= \int \mathrm{d}^d y[-g(y)]^{\frac{1}{2}} \int \mathrm{d}^d x[-g(x)]^{\frac{1}{2}} \\ &\quad \times K_{F\ \nu\sigma}(f;y)\mu^{4-d}\bar{\omega}^{\sigma\rho}(y;x)K_{F\ \rho\mu}(x;i) \qquad (4.3) \\ &= -(-ie)^2\mu^{4-d}\int \mathrm{d}^d y[-g(y)]^{\frac{1}{2}} \int \mathrm{d}^d x[-g(x)]^{\frac{1}{2}} \\ &\quad \times K_{F\ \nu\sigma}(f;y)\mathrm{Tr}\left[\gamma^\sigma S_F(y;x)\gamma^\rho S_F(x;y)\right] K_{F\ \rho\mu}(x;i) \end{aligned}$$

most easily evaluated in the embedding space $\mathcal{M}$ (see eqns. (2.25) and (3.29)):

$$\begin{aligned} &= (-ie)^2\mu^{4-d}\int_{\mathcal{N}} \mathrm{d}^d x \int_{\mathcal{N}} \mathrm{d}^d y K_{F\ NQ}(f;y) \\ &\quad \times \mathrm{Tr}\left[\bar{\gamma}^Q \hat{\not{y}} S_F(y;x)\bar{\gamma}^P \hat{\not{x}} S_F(x;y)\right] K_{F\ PM}(x;i). \qquad (4.4) \end{aligned}$$

where the x and y integrals run over the space $\mathcal{N}$ embedded in the $d+1$-dimensional space $\mathcal{M}$. As in Minkowski space, this diagram contains a divergent part, which should be taken care of by regularization and subsequent renormalization. How to go about that in coordinate space is outlined in Collins [Col84] chapter 11.

A first examination of the degree of divergence encountered here reveals no surprises compared to that exhibited by the first order photon propagator correction in Minkowski space. The spinor propagator pole $S_F \sim (-\lambda)^{(1-d)/2} \sim (x-y)^{1-d}$ (from 3.38 and A.5) suggests, in the $d \to 4$ limit, a superficial quadratic divergence to the diagram as the points x and y coincide. This close-proximity divergence is the configuration-space guise of the high-momentum divergence encountered in more familiar momentum-space treatments in Minkowski space. However, the Ward identity – which is basically a local identity which can be trivially proven to be valid in anti-de Sitter space, with the obvious replacement of partial derivatives by covariant derivatives – dictates that vector propagator corrections be transverse

$$i\mathcal{D}_\nu(y)\bar{\omega}^{\nu\mu}(y;x) = 0$$

and therefore, in the operator formulation used below, to be proportional to

$$g^{\nu\mu}(i\mathcal{D})^2 - i\mathcal{D}^\nu i\mathcal{D}^\mu + \mathcal{R}^{\nu\mu}.$$

This will effectively extract two orders out of the integration, leaving just a logarithmic divergence. This is made explicit in the computation below.

Extracting the divergence out of (4.4) means one is interested in the pole part and thus the small $x - y \stackrel{\text{def}}{=} \varepsilon$ behavior of the x integral. A convenient way to go about that is to remove the x dependence in $K_F(x;i)$ by rewriting $\bar{\omega}^{QP}(y;x)$ as an operator acting on the propagator $K_F(y;i)$[3]. That way the latter can be pulled out of the x integral:

$$\begin{aligned}
&\int \mathrm{d}^d x\ \omega^{QP}(y;x)K_F(x;i) \\
&= \int \mathrm{d}^d x\ \omega^{QP}(y;x) \\
&\qquad \times \sum_{n=0}^{\infty} \frac{1}{n!}(x-y)^{M_1}\cdots(x-y)^{M_n} \times \frac{\partial^n}{\partial y^{M_1}\cdots\partial y^{M_n}} K_F(y;i) \\
&= \left[\sum_{n=0}^{\infty}\int \mathrm{d}^d x\ \omega^{QP}(y;x)\frac{R^n}{n!}\hat{\varepsilon}^{M_1}\cdots\hat{\varepsilon}^{M_n} \times \frac{\partial^n}{\partial y^{M_1}\cdots\partial y^{M_n}}\right] K_F(y;i) \\
&\stackrel{\text{def}}{=} \hat{\omega}^{QP}(y)K_F(y;i)
\end{aligned}$$

for sufficiently well-behaved K_F. The operator $\hat{\omega}^{QP}(y)$, then, is the configuration space equivalent of the truncated diagram.

Computation

In order to obtain the pole part of (4.4), $\hat{\omega}^{QP}(y)$ will have to be evaluated only up to order $(x-y)^2\partial_x^2$ or, in terms of the embedding space $\mathcal{M}$, up to

[3]In the Minkowski case of course, momentum space instead of configuration space can be used, leading to a simple multiplication instead of a cumbersome operator. Unfortunately, the Anti-de Sitter equivalent of momentum space, while feasible, turned out to be nowhere near as convenient. Since in the general curved space case, momentum is ill defined, configuration space is used throughout.

order $\lambda\mathcal{D}_x^2$. Then the divergent terms in the trace need to be worked out and the resultant integral tamed using dimensional regularization.

$$
\begin{aligned}
&\operatorname*{Pole}_{d\to 4}\ \hat{\omega}^{QP}(y) \\
&= \ (-ie)^2 \operatorname*{Pole}_{d\to 4} \int_{\mathcal{N}} \mathrm{d}^d x \ \mathrm{Tr}\left[\bar{\gamma}^Q \hat{y}\!\!\!/ S_F(y;x)\gamma^P \hat{x}\!\!\!/ S_F(x;y)\right] \\
&\qquad \times \left[1 + R\hat{\varepsilon}^M \partial_M + \frac{1}{2}R^2 \hat{\varepsilon}^M \hat{\varepsilon}^N \partial_M \partial_N + \ldots\right]
\end{aligned} \tag{4.5}
$$

(see B.3 for more details). The trace part is, using $\hat{y}\!\!\!/ S_F(y;x) = [R\partial\!\!\!/ + m'\hat{y}\!\!\!/\ ]G_F(\lambda(y;x))$ (3.38):

$$
\begin{aligned}
&\mathrm{Tr}\left[\bar{\gamma}^Q \hat{y}\!\!\!/ S_F(y;x)\bar{\gamma}^P \hat{x}\!\!\!/ S_F(x;y)\right] \\
&= \ N\left[\eta^{QS}\eta^{PR} - \eta^{QP}\eta^{SR} + \eta^{QR}\eta^{PS}\right] \\
&\qquad \times \left([R\partial_{yS} + m'\hat{y}_S]G_F(\lambda(y;x))\right) \\
&\qquad \times \left([R\partial_{xR} + m'\hat{x}_R]G_F(\lambda(x;y))\right)
\end{aligned}
$$

where $N \stackrel{\text{def}}{=} |\mathrm{Tr}\bar{\gamma}_Q^2|$, the dimensionality of the gamma matrices. Although formally speaking there is certainly no smooth $d \to 4$ limit for anticommuting γ matrices, as has been argued in section 3.2 this is of no import in the embedding problem. Therefore the entire issue can be side-stepped here and it will be merely assumed that, in the end of things, $N_{d=4} = 4$.

Multiplying out all these terms gives contributions proportional to $G_F'^2(\lambda)$, $G_F'(\lambda)G_F(\lambda)$ and $G_F^2(\lambda)$, the primes denoting the derivative with respect to $\lambda(x;y) = \lambda(y;x)$. For the sake of clarity they will be examined individually:

- Working out the $\partial_S G_F(\lambda)\ \partial_R G_F(\lambda)$ term

  $$\mathrm{Tr}\left[\bar{\gamma}^Q (R\partial\!\!\!/_y G_F(\lambda))\gamma^P (R\partial\!\!\!/_x G_F(\lambda))\right]$$

  gives, using $R\partial_{yS}\lambda = -2z(\hat{x}_S - z\hat{y}_S)$ (2.37),

  $$
  \begin{aligned}
  = \ 4Nz^2 \Big(&(\hat{x}^Q - z\hat{y}^Q)(\hat{y}^P - z\hat{x}^P) \\
  &- (\hat{x}_S - z\hat{y}_S)(\hat{y}^S - z\hat{x}^S)\eta^{PQ} \\
  &+ (\hat{x}^P - z\hat{y}^P)(\hat{y}^Q - z\hat{x}^Q)\Big)\, G_F'^2(\lambda).
  \end{aligned}
  $$

  The trace as a whole is sandwiched between $K(f;y)$ and $K(x;i)$. Since these are tensors pushed up from $\mathcal{N}$ to $\mathcal{M}$ they are in the kernel of

$T(y, \mathcal{M}/\mathcal{N}) \otimes T(x, \mathcal{M}/\mathcal{N})$ (2.13): $K_{NQ}(f; y)y^Q = x^P K_{PM}(x; i) = 0$. The latter also follows directly from the structure (3.56) of these propagators. Thus, the trace effectively reduces to

$$\begin{aligned} &\stackrel{\text{eff}}{=} 4N(1-\lambda)\left((2-\lambda)\hat{x}^Q\hat{y}^P + z\lambda P^{QP}(y)\right) G_F'^2(\lambda) \\ &= 4N\left((-2+3\lambda-\lambda^2)\hat{\varepsilon}^Q\hat{\varepsilon}^P + z(\lambda-\lambda^2)P^{QP}(y)\right) G_F'^2(\lambda). \end{aligned}$$

The $d \to 4$ pole part of $G_F'^2$ can be read from the propagators in Appendix A.

- The mixed $\partial_S G_F(\lambda) G_F(\lambda)$ and $G_F(\lambda)\partial_R G_F(\lambda)$ terms both effectively yield the same contribution:

$$\begin{aligned} &\mathrm{Tr}\left[\bar{\gamma}^Q (R\partial\!\!\!/_y G_F(\lambda))\gamma^P(m'\hat{x}\!\!\!/ G_F(\lambda))\right] \\ &= -2Nm'z\left((\hat{x}^Q - z\hat{y}^Q)\hat{x}^P - (\hat{x}_S - z\hat{y}_S)\hat{x}^S\eta^{QP}\right. \\ &\qquad \left. +(\hat{x}^P - z\hat{y}^P)\hat{x}^Q\right) G_F'(\lambda)G_F(\lambda) \\ &\stackrel{\text{eff}}{=} -2Nm'z\left(-\lambda P^{QP}(y) + z\hat{\varepsilon}^Q\hat{\varepsilon}^P\right) G_F'(\lambda)G_F(\lambda). \end{aligned}$$

- Finally the $G_F^2(\lambda)$ contribution reads

$$\begin{aligned} &\mathrm{Tr}\left[\bar{\gamma}^Q(m'\hat{y}\!\!\!/ G_F(\lambda))\gamma^P(m'\hat{x}\!\!\!/ G_F(\lambda))\right] \\ &= Nm'^2\left(\hat{y}^Q\hat{x}^P - \hat{y}\cdot\hat{x}\eta^{QP} + \hat{y}^P\hat{x}^Q\right) G_F^2(\lambda) \\ &\stackrel{\text{eff}}{=} Nm'^2\left(-\hat{\varepsilon}^Q\hat{\varepsilon}^P - zP^{QP}(y)\right) G_F^2(\lambda). \end{aligned}$$

Taking these terms together, expanding $\hat{x}\cdot\hat{y} = z = \sqrt{1-\lambda}$ in λ and retaining only the orders actually contributing to the divergence,

$$\begin{aligned} &\mathrm{Tr}\left[\bar{\gamma}^Q\hat{y}\!\!\!/ S_F(y;x)\bar{\gamma}^P\hat{x}\!\!\!/ S_F(x;y)\right] \\ &= 2N\left((-4+6\lambda-2\lambda^2)\hat{\varepsilon}^Q\hat{\varepsilon}^P + (2\lambda-3\lambda^2)P^{QP}(y)\right) G_F'^2(\lambda) \\ &\quad -4Nm'\left(\hat{\varepsilon}^Q\hat{\varepsilon}^P - \lambda P^{QP}(y)\right) G_F'(\lambda)G_F(\lambda) \\ &\quad -Nm'^2 P^{QP}(y)G_F^2(\lambda) + \ldots \end{aligned} \tag{4.6}$$

The singular behavior of the $G_F(\lambda)$ and $G_F'(\lambda)$ can be extracted from Eqns. (A.2) and (A.5); the most divergent term is $\sim \lambda^{1-d/2} \sim \varepsilon^{2-d}$ for $G_F(\lambda)$ and $\sim \lambda^{-d/2} \sim \varepsilon^{-d}$ for $G_F'(\lambda)$. The trace, then, when y is integrated over the

d-dimensional Anti-de Sitter manifold $\mathcal{N}$ in the embedding space space $\mathcal{M}$, exhibits a quadratic singularity in the $G_F'^2(\lambda)$ part and logarithmic singularity in the others. To be precise, after substituting the parts of $G_F(\lambda)$ and $G_F'(\lambda)$ contributing to the singularities and working things out, one finds

$$\begin{aligned} = \; & \frac{N\Gamma(\frac{d}{2})^2}{4\pi^d R^{2d-4}}\Big[2(-\lambda_F)^{-d}\hat{\varepsilon}^Q\hat{\varepsilon}^P \\ & - \frac{2(m'-1)(m'+d-1)+d}{d-2}(-\lambda_F)^{1-d}\hat{\varepsilon}^Q\hat{\varepsilon}^P \\ & - \frac{2(d-3)(m'-1)(m'+d-1)+d(d-4)+2}{2(d-2)^2} \\ & \times(-\lambda_F)^{2-d}P^{QP}(y) + (-\lambda_F)^{1-d}P^{QP}(y) + \ldots\Big]. \end{aligned}$$

Now the truncated diagram (4.5) can be computed in full. The dimensional regularization of the diverging $\int_{\mathcal{N}} \mathrm{d}^4x$ integral has turned the $x \to y$ singularities into $d \to 4$ singularities; the details of the procedure are relegated to Appendix B. The result is

$$\begin{aligned} & \text{Pole } \hat{\omega}^{QP}(y) \\ = \; & -i(-ie)^2 \frac{N\Gamma\left(\frac{d}{2}\right)}{4\pi^{\frac{d}{2}}R^{d-4}} \\ & \times\Bigg\{2\left[\frac{1}{d}P^{QP}(y)(1-R\hat{y}\cdot\partial) + \frac{1}{2}\hat{y}^Q\hat{y}^P - \frac{R}{d}\hat{y}^{\{Q}\mathcal{D}^{P\}}\right. \\ & \left. + \frac{R^2}{d(d+2)}\Big(P^{QP}(y)P^{KL}(y) + 2P^{QK}(y)P^{PL}(y)\Big)\partial_K\partial_L\right] \\ & + \frac{2(m'-1)(m'+d-1)+d}{d-2}\cdot\frac{2}{d}P^{QP}(y) \\ & - \frac{2(d-3)(m'-1)(m'+d-1)+d(d-4)+2}{(d-2)^2}P^{QP}(y) \\ & + P^{QP}(y)\left[-(1-R\hat{y}\cdot\partial) - \frac{R^2}{d}P^{KL}(y)\partial_K\partial_L\right]\Bigg\}\frac{1}{4-d} \end{aligned}$$

Since this is to be contracted with $K_{NQ}(f;y)$, all terms proportional to y^Q vanish:

$$\begin{aligned} \overset{\text{eff}}{=} \; & -i(-ie)^2 \frac{N\Gamma\left(\frac{d}{2}\right)}{4\pi^{\frac{d}{2}}R^{d-4}} \\ & \times\Bigg\{-\frac{R^2}{d+2}\Big(P^{QP}(y)P^{KL}(y) - \frac{4}{d}P^{QK}(y)P^{PL}(y)\Big)\partial_K\partial_L \\ & - \frac{2(d-4)}{d(d-2)^2}\Big((d-1)(m'-1)(m'+d-1) \end{aligned}$$

$$
\begin{aligned}
&+ d^2 - 2d + 2\Big) P^{QP}(y) \\
&+ \frac{2(d-3)}{(d-2)^2} P^{QP}(y) + \frac{d-2}{d} P^{QP}(y) R\hat{y}\cdot\partial \Big\} \frac{1}{4-d}
\end{aligned}
$$

There is something disconcerting about this expression, as it appears to violate the Ward identity in curved space. In other words, the correction is not purely transverse. However, all violating terms between the square brackets are proportional to $d-4$ and therefore do in fact belong to the finite part. Since a great number of finite terms have already been thrown away, it is no surprise that the remainder violates the Ward identity; indeed, these terms should properly be left out as well, giving the rather more appetizing

$$
\begin{aligned}
&\operatorname*{Pole}_{d\to 4}\ \hat{\omega}^{QP}(y) \\
&= -i(-ie)^2 \frac{N\Gamma\left(\frac{d}{2}\right)}{4\pi^{\frac{d}{2}} R^{d-4}} \\
&\quad \times \left\{ -\frac{R^2}{6} \left(P^{QP}(y) P^{KL}(y) - P^{QK}(y) P^{PL}(y) \right) \partial_K \partial_L \right. \\
&\quad \left. + \frac{1}{2} P^{QP}(y)(1 + R\hat{y}\cdot\partial) \right\} \frac{1}{4-d}
\end{aligned} \tag{4.7}
$$

which *does* satisfy the Ward identity (or rather its equivalent in $\mathcal{M}$):

$$
\operatorname*{Pole}_{d\to 4}\ \mathcal{D}_Q \hat{\omega}^{QP}(y) = 0 \tag{4.8}
$$

as can be verified by direct computation. Note that the very last (curvature related) term in (4.7) is essential to get the necessary cancellation.

Results and discussion

Equation (4.7) can be rewritten in terms of the covariant derivative $\mathcal{D}$ which directly maps to the covariant derivative in the embedded space $\mathcal{N}$ (2.24, 2.31):

$$
\begin{aligned}
&= -i(-ie)^2 \frac{N\Gamma\left(\frac{d}{2}\right)}{4\pi^{\frac{d}{2}} R^{d-4}} \\
&\quad \times \left\{ -\frac{R^2}{6} \left(P^{QP}(y)\mathcal{D}^2 - \mathcal{D}^Q \mathcal{D}^P \right) + \frac{1}{2} P^{QP}(y) \right\} \frac{1}{4-d}.
\end{aligned} \tag{4.9}
$$

Taking the $d \to 4$ limit and translating back to the embedded space,

$$
\omega^{\rho\sigma}(x) = i\frac{e^2}{6\pi^2} \left\{ \left(g^{\sigma\rho}(i\mathcal{D})^2 - i\mathcal{D}^\rho i\mathcal{D}^\sigma \right) + \mathcal{R}^{\rho\sigma} \right\} \frac{1}{4-d}. \tag{4.10}
$$

The factor R^2 dropped out due to the definition of the spinor propagator (3.37). The last term proportional to the Ricci curvature tensor $\mathcal{R}^{\rho\sigma}$ (2.5) is a pure curvature contribution. The expression as a whole is highly similar to the textbook result for Minkowski space[4]. The flat space limit reproduces it exactly, as it should. Compared to the Minkowski-space result, the contribution of the curvature is minimal in the sense that the new terms are exactly those needed to satisfy the covariant version of the Ward identity. However, one can see from the preceding calculation that a great many extra terms crop up for nonzero curvature or, equivalently, finite R. To be precise, all terms proportional to R^k with $k < 2$ are pure curvature terms. Most of them happen to belong to the finite part in this calculation, but since there appears to be no a priori reason why they should cancel, this suggests that the finite result will have a rich curvature dependent structure. Unfortunately, as discussed in a little more depth in Appendix B.2, the method used is not very good at evaluating the finite part of Feynman diagrams (a fact already remarked upon by Collins [Col84] p.286) and a different approach needs to be found to obtain exact results.

Finally, it is worth stating that the result corresponds to a photon field renormalization constant

$$Z_3 = 1 + \frac{2\alpha}{3\pi}\frac{1}{d-4}. \tag{4.11}$$

In line with that derived by Drummond and Shore [DS79] for spherical spacetime, this is equal to the flat space renormalization.

[4] see for example Le Bellac [Bel91] 12.3.19:

$$\omega^{\rho\sigma}(q^2) = -i\frac{e^2}{6\pi^2}\left[g^{\rho\sigma}q^2 - q^\rho q^\sigma\right]\frac{1}{4-d} + \mathcal{O}((4-d)^0)$$

The overall sign difference stems from the slightly different conventions used.

4.2 Electron self-energy

Consider the lowest order electron self-energy diagram shown in figure 4.3. The Feynman rules from the introduction give the following expression for

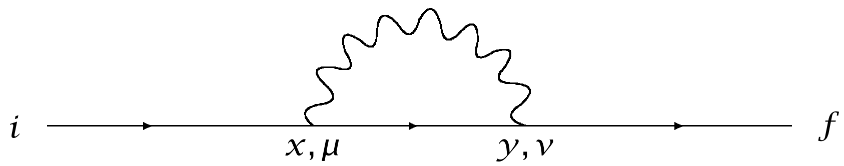


figure 4.3: The self-energy diagram.

the one-loop contribution to the electron propagator:

$$\begin{aligned} S_F^{[1]}(f;i) &= \int \mathrm{d}^d y[-g(y)]^{\frac{1}{2}} \int \mathrm{d}^d x[-g(x)]^{\frac{1}{2}} \\ &\qquad \times S_F(f;y)\Big(-i\Sigma(y;x)\Big)S_F(x;i) \\ &= (-ie)^2 \int \mathrm{d}^d y[-g(y)]^{\frac{1}{2}} \int \mathrm{d}^d x[-g(x)]^{\frac{1}{2}} \\ &\qquad \times S_F(f;y)\gamma^\nu S_F(y;x)\gamma^\mu K_{\nu\mu}(y;x)S_F(x;i). \end{aligned} \tag{4.12}$$

As was the case with last section's vacuum bubble, the $y - x$ dependence of $S_F(y;f)$, together with the x integral, can be absorbed in $\Sigma(y;x)$ by turning the latter into an operator:

$$= \int \mathrm{d}^d y S_F(f;y)\Big(-i\hat{\Sigma}(y)\Big)S_F(y;i)$$

where the truncated diagram $\hat{\Sigma}$ is

$$\hat{\Sigma}(y) \overset{\mathrm{def}}{=} \sum_{n=0}^{\infty} \int \mathrm{d}^d x \; \Sigma(y;x)\frac{1}{n!}(x-y)^n \times \frac{\partial^n}{\partial y^n} \tag{4.13}$$

A convenient way to evaluate the y integral is to embed it in the $d+1$-dimensional embedding space $\bar{\mathcal{M}}$:

$$\begin{aligned} = \; -ie^2 \sum_{n=0}^{\infty} \int_{\mathcal{N}} & \mathrm{d}^d x P^{N}{}_{Q}(y)\bar{y}^Q \not{\hat{y}} S_F(y;x) \\ & \times \not{\hat{x}} P^{M}{}_{P}(x)\bar{y}^P K_{NM}(y;x)\frac{R^n}{n!}\hat{\varepsilon}^{M_1}\cdots\hat{\varepsilon}^{M_n} \times \frac{\partial^n}{\partial y^{M_1}\cdots\partial y^{M_n}} \end{aligned}$$

$P^N{}_Q(y)$, the projection operator defined in (2.20), is $g^\nu{}_\sigma(y_\mathcal{N})$ embedded in $\bar{\mathcal{M}}$: it is the equivalent in $\bar{\mathcal{M}}$ of summing μ, ν over the d dimensions of $\mathcal{N}$ at points y and x, respectively.

Just like its Minkowski analogue, this diagram is divergent. Both the spinor and the vector propagator are singular as the separation between x and y tends to zero or, equivalently, for increasing loop momentum. The spinor propagator $S_F \overset{\text{eff}}{\sim} (-\lambda)^{(1-d)/2}$ (3.38, A.5) and the vector propagator, given (3.80),

$$\begin{aligned} K_{F\ NM}(y;x) \quad \sim \quad & P_N(y)\cdot P_M(x)(-\lambda_F(y;x))^{(2-d)/2} \\ & + P_N(y)\cdot \hat{x}\ \hat{y}\cdot P_M(x)(-\lambda_F(x,y))^{-d/2} + \mathcal{O}(\lambda^{4-d}) \end{aligned}$$

Thus the superficial degree of divergence for the integral as a whole, with the integrand proportional to $(-\lambda)^{3/2-d}$, is $3-d$ — and just like its more familiar counterpart, its actual divergence will turn out to be an order lower, i.e. logarithmic in the $d \to 4$ limit, with the result being proportional to kinematic terms $i\not{\mathcal{D}}$ and m_e, and as it turns out, the curvature $\mathcal{R}$.

From the structure of $K_{F\ NM}(y;x)$ quoted above it can also be seen that $\hat{\Sigma}(y;x)$ simplifies to

$$\begin{aligned} = \quad & -ie^2 \sum_{n=0}^{\infty} \int_{\mathcal{N}} \mathrm{d}^d x \bar{y}^N \hat{\not{y}} S_F(y;x) \hat{\not{x}} \bar{y}^M K_{F\ NM}(y;x) \\ & \times \frac{R^n}{n!} \hat{\varepsilon}^{M_1} \cdots \hat{\varepsilon}^{M_n} \times \frac{\partial^n}{\partial y^{M_1} \cdots \partial y^{M_n}} \end{aligned} \tag{4.14}$$

i.e. the projection operators P can be absorbed in the vector propagator K_F.

Computation

Massless fields in adS have peculiar properties [FF78, FF80b]. However, the massless vector field that corresponds to the flat space electrodynamic field in the Inönü-Wigner contraction can be obtained by simply taking the massive vector field in the massless limit [JD87]. The expressions derived in section 3.3 can therefore be used here. Furthermore, if one carefully takes the limit to flat space, the massless 4-dimensional anti-de Sitter electrodynamic field gives either positive or negative helicity states but not both. As Binegar et al noted, normal QED can be obtained by the introduction of two dynamically independent vector potentials that lose their independence in the flat space limit [BFH83]. This distinction can actually be realized at the propagator level [Gaz85]. However, the d-dimensional propagator (3.85) makes no such distinctions and will propagate both helicities. The loss of conformal invariance for $d \neq 4$ ensures that anomalous mass terms remain even in the massless case; by *first* fully evaluating the diagram of fig. 4.3 in d-dimensional CadS and *then* taking the flat space and $d \to 4$ limits, both helicity contributions are incorporated and the full Minkowski space physics is recovered.

For a computation of the divergent part, the first few terms of the summation in Eqn. (4.14) will suffice. Working out first the $S_F K_{F\ NM}$ part in terms of the scalar propagator G_F and the components of K_F using (3.38) and (3.56) respectively, four terms arise:

$$\begin{aligned} & R\bar{\gamma}^N \hat{\not{y}} S_F(y;x) \hat{\not{x}} \bar{\gamma}^M K_{NM}(y;x) \\ = \;& \bar{\gamma}^N \left([R\not{\partial}_y + m_e' \hat{\not{y}}]\, G_F(\lambda(y;x)) \right) \\ & \times \hat{\not{x}} \bar{\gamma}^M \left[P_N(y) \cdot P_M(x) \mathcal{F}_F(\lambda(y;x)) \right. \\ & \quad \left. + P_N(y) \cdot \hat{x}\ \hat{y} \cdot P_M(x) \mathcal{G}_F(\lambda(y;x)) \right] \end{aligned}$$

The two terms each in the $S_F(\lambda)$ and $K_{F\ NM}(\lambda)$ parts can be multiplied out to give four terms. Some cumbersome if straightforward computational exercise leads to the following expressions for these:

- The first, $\not{\partial}_y G_F(\lambda)\mathcal{F}_F(\lambda)$ term, where $R\partial_{yS}\lambda = -2z(\hat{x}_S - z\hat{y}_S)$ (2.37), boils down to

$$-2z\bar{\gamma}^N(\hat{\not{x}} - z\hat{\not{y}})\hat{\not{x}}\bar{\gamma}^M P_N(y) \cdot P_M(x) \mathcal{F}(\lambda) G_F'(\lambda).$$

  Now equation (2.29) for P_{MN} can be substituted in the $P(y) \cdot P(x)$s and the $\bar{\gamma}$ algebra (3.29) used to simplify the resultant expression:

$$\begin{aligned} = \;& -2z \Big(\bar{\gamma}^N \bar{\gamma}_N - z\bar{\gamma}^N \hat{\not{y}}\hat{\not{x}}\bar{\gamma}_N - \hat{\not{x}}(\hat{\not{x}} - z\hat{\not{y}})\hat{\not{x}}\hat{\not{x}} \\ & -\hat{\not{y}}(\hat{\not{x}} - z\hat{\not{y}})\hat{\not{x}}\hat{\not{y}} + z\hat{\not{y}}(\hat{\not{x}} - z\hat{\not{y}})\hat{\not{x}}\hat{\not{x}} \Big) \mathcal{F}_F(\lambda) G_F'(\lambda) \\ = \;& -2z \Big((d-1) + (d-2)z - (2d-3)z^2 \\ & +(d-2)z\hat{\not{x}}\hat{\not{y}} \Big) \mathcal{F}_F(\lambda) G_F'(\lambda) \end{aligned}$$

- The $\not{\partial}_y G_F(\lambda)\mathcal{G}_F(\lambda)$ term evaluates to

$$\begin{aligned} & -2z\bar{\gamma}^N(\hat{\not{x}} - z\hat{\not{y}})\hat{\not{x}}\bar{\gamma}^M P_N(y) \cdot \hat{x}\ \hat{y} \cdot P_M(x) \mathcal{G}_F(\lambda) G_F'(\lambda) \\ = \;& -2z \Big(\hat{\not{x}}(\hat{\not{x}} - z\hat{\not{y}})\hat{\not{x}}\hat{\not{y}} - 2z + 2z^2 \hat{\not{x}}\hat{\not{y}} + z^2 \hat{\not{y}}\hat{\not{x}} - z^3 \Big) \\ & \times \mathcal{G}_F(\lambda) G_F'(\lambda) \\ = \;& -2z \left(\lambda\hat{\not{x}}\hat{\not{y}} + \lambda - \lambda z \right) \mathcal{G}_F(\lambda) G_F'(\lambda). \end{aligned}$$

- Thirdly, the $G_F(\lambda)\mathcal{F}_F(\lambda)$ part yields

$$\begin{aligned} & m_e' \bar{\gamma}^N \hat{y}\!\!\!/ \hat{x}\!\!\!/ \bar{\gamma}^M P_N(y) \cdot P_M(x) \mathcal{F}_F(\lambda) G_F(\lambda) \\ &= m_e' \left(\bar{\gamma}^M \hat{y}\!\!\!/ \hat{x}\!\!\!/ \bar{\gamma}_M - 2\hat{x}\!\!\!/ \hat{y}\!\!\!/ + z \right) \mathcal{F}_F(\lambda) G_F(\lambda) \\ &= m_e' \left(-(d-1)\hat{\varepsilon}\!\!\!/ \hat{y}\!\!\!/ - (d-1) \right. \\ &\qquad \left. +(2d-1)z \right) \mathcal{F}_F(\lambda) G_F(\lambda) \end{aligned}$$

- while last and least, the $G_F(\lambda)\mathcal{G}_F(\lambda)$ contribution,

$$\begin{aligned} & m_e' \bar{\gamma}^N \hat{y}\!\!\!/ \hat{x}\!\!\!/ \bar{\gamma}^M P_N(y) \cdot \hat{x}\ \hat{y} \cdot P_M(x) G_F(\lambda) \mathcal{G}_F(\lambda) \\ &= m_e' \left(\hat{x}\!\!\!/ \hat{y}\!\!\!/ \hat{x}\!\!\!/ \hat{y}\!\!\!/ - 2z\hat{x}\!\!\!/ \hat{y}\!\!\!/ + z^2 \right) G_F(\lambda) \mathcal{G}_F(\lambda) \\ &= m_e' \left(-\lambda \right) G_F(\lambda) \mathcal{G}_F(\lambda). \end{aligned}$$

Adding these terms together, expanding $\hat{x} \cdot \hat{y} = z = \sqrt{1-\lambda}$ in λ retaining only those parts contributing to the overall divergence of the integral, and substituting the contributing terms in the expansions of $\mathcal{F}_F(\lambda)$ and $\mathcal{G}_F(\lambda)$ (A.14) — which are the highest order terms only, as the resultant integration diverges only logarithmically:

$$\begin{aligned} \mathcal{F}_F(\lambda) &= -\frac{\Gamma\left(\frac{d-2}{2}\right)}{8i\pi^{\frac{d}{2}} R^{d-2}} \frac{1+c}{c} (-\lambda_F)^{\frac{2-d}{2}} + \ldots \\ \mathcal{G}_F(\lambda) &= +\frac{\Gamma\left(\frac{d-2}{2}\right)^2}{4i\pi^{\frac{d}{2}} R^{d-2}} \frac{1-c}{c} (-\lambda_F)^{-\frac{d}{2}} + \ldots \end{aligned} \tag{4.15}$$

the sum of those parts contributing to the divergence emerges as

$$\begin{aligned} & R\bar{\gamma}^N \hat{y}\!\!\!/ S_F(y;x) \hat{x}\!\!\!/ \bar{\gamma}^M K_{F\ NM}(y;x) \\ &= -\frac{\Gamma\left(\frac{d-2}{2}\right)}{4i\pi^{\frac{d}{2}} R^{d-2}} \\ &\quad \times \Bigg\{ \left(-(d-2)\frac{1+c}{c} - (d-2)\frac{1-c}{c} + \ldots \right) (-\lambda_F)^{\frac{2-d}{2}} \hat{\varepsilon}\!\!\!/ \hat{y}\!\!\!/ G_F'(\lambda) \\ &\qquad + \left(\frac{3d-4}{2}\ \frac{1+c}{c} + \frac{d-2}{2}\ \frac{1-c}{c} + \ldots \right) (-\lambda_F)^{\frac{4-d}{2}} G_F'(\lambda) \\ &\qquad - m_e' \left(-\frac{d}{2}\ \frac{1+c}{c} + \frac{d-2}{2}\ \frac{1-c}{c} + \ldots \right) (-\lambda_F)^{\frac{2-d}{2}} G_F(\lambda) \Bigg\} \end{aligned}$$

The $\hat{\varepsilon}\!\!\!/ \hat{y}\!\!\!/$ can be split into parts odd and even in $\hat{x}$ (B.5):

$$\hat{\varepsilon}\!\!\!/ \hat{y}\!\!\!/ = (P_{MN}(y) + \hat{y}_M \hat{y}_N) \hat{\varepsilon}^M \bar{\gamma}^N \hat{y}\!\!\!/ = \hat{\varepsilon} \cdot P\!\!\!/(y) \hat{y}\!\!\!/ - \frac{1}{2}\hat{\varepsilon}^2$$

giving[5]

$$= -\frac{\Gamma\left(\frac{d-2}{2}\right)}{4i\pi^{\frac{d}{2}}R^{d-2}}\Big\{-2\frac{d-2}{c}(-\lambda_F)^{\frac{2-d}{2}}\hat{\varepsilon}\cdot P\!\!\!/(y)\hat{y}\!\!\!/G'_F(\lambda)$$
$$+ m'_e\Big[d-1+\frac{1}{c}\Big](-\lambda_F)^{\frac{2-d}{2}}G_F(\lambda)$$
$$+ (d-1)\frac{1+c}{c}(-\lambda_F)^{\frac{4-d}{2}}G'_F(\lambda) + \dots\Big\}$$

Now it is time to go back to the full expression $\hat{\Sigma}(y;x)$ for the truncated diagram (4.14) and expand the contributing (first) order of $G_F(\lambda)$ (eqn. A.2) and $G'_F(\lambda)$ (A.5). Only the first two orders in the Taylor expansion operator contribute to the singular part:

$$\operatorname*{Pole}_{d\to 4}\hat{\Sigma}(y)$$
$$= -\frac{\Gamma\left(\frac{d-2}{2}\right)}{4i\pi^{\frac{d}{2}}R^{d-2}}\,\frac{e^2\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}R^{d-2}}\int_{\mathcal{N}}\mathrm{d}^d x R^{-1}$$
$$\times\Big\{-2\frac{d-2}{c}\,\frac{d-2}{2}(-\lambda_F)^{1-d}\hat{\varepsilon}\cdot P\!\!\!/(y)\hat{y}\!\!\!/R\hat{\varepsilon}\cdot\partial$$
$$+ m'_e\Big[d-1+\frac{1}{c}\Big](-\lambda_F)^{2-d}$$
$$+ (d-1)\frac{1+c}{c}\,\frac{d-2}{2}(-\lambda_F)^{2-d}\Big\}$$

since the odd $\hat{\varepsilon}\cdot P\!\!\!/(y)\hat{y}\!\!\!/$ term contributes to the pole only when multiplied with $R\hat{\varepsilon}\cdot\partial$, giving a logarithmically diverging integral (B.9) while for the remaining terms it is the other way around, giving further logarithmic divergences (B.4). The integration is understood to run over all the sheets k of the covering space, introducing a phase factor on sheets $k \neq 0$ which will however not contribute to the pole, as shown in section B.2. Upon performing the integration for $d = 4 + \varepsilon$, the following result emerges:

$$= \frac{\Gamma\left(\frac{d-2}{2}\right)}{4i\pi^{\frac{d}{2}}R^{d-2}}ie^2\frac{1}{2}\Gamma\left(\frac{d-2}{2}\right)R$$

[5]This expression can be directly compared to its flat space equivalent in the $c = 1$ Feynman gauge. A straightforward configuration space computation in Minkowski space yields:

$$-i2(d-2)\frac{1}{2\pi}\mathrm{Im}\,(\frac{\not{x}}{\lambda_{-\epsilon}})G' + m_e\frac{1}{2\pi}\mathrm{Im}\,(\frac{d}{\lambda_{-\epsilon}})G$$

The only significant difference in the anti-de Sitter case is the addition of a curvature term.

$$\times \left\{-(d-1)\left(-m_e' - \frac{d-2}{2}\frac{1+c}{c}\right) - \frac{1}{c}\left(\frac{4}{d}\left(\frac{d-2}{2}\right)^2 R\hat{y}\!\!\!/\mathcal{D}\!\!\!\!/ - m_e'\right)\right\}\frac{1}{4-d}$$
$$= \frac{e^2}{8\pi^2 R}\left\{3m_e R - \frac{1}{c}\left[y\!\!\!/\mathcal{D}\!\!\!\!/ - m_e R - 2\right]\right\}\frac{1}{4-d} + \text{ finite terms.} \tag{4.16}$$

using $m_e' = m_e R - d/2 + 1 \overset{d\to 4}{=} m_e R - 1$ (3.33).

Results and discussion

Translating (4.16) back to the embedded space $\mathcal{N}$ where (3.30) in the natural vielbein means that $\hat{y}\!\!\!/\mathcal{D}\!\!\!\!/$ becomes $i\mathcal{D}\!\!\!\!/ + \frac{d}{2}$, the final result emerges:

$$\underset{d\to 4}{\text{Pole}}\,\hat{\Sigma}(y) = \frac{e^2}{8\pi^2}\left\{3m_e - \frac{1}{c}\left[i\mathcal{D}\!\!\!\!/ - m_e\right]\right\}\frac{1}{4-d} \tag{4.17}$$

This is precisely the structure of the textbook result[6] with the obvious substitution of the covariant derivative $i\mathcal{D}\!\!\!\!/$ for the momentum $p\!\!\!/$. In the $R\to\infty$ contraction to Minkowski space it is completely identical to it. As was the case with the vacuum polarization diagram, despite all the curvature dependencies of the intermediate results, their contributions to the singularity cancel out. It is very unfortunate that the calculational technique employed is not powerful enough to compute curvature-dependent finite parts (Appendix B.2, [Col84] p.286).

The result is also analogous to the flat space case in that it represents a gauge-independent mass scaling

$$\delta m_e = m_e \frac{3\alpha}{2\pi}\frac{1}{d-4} \tag{4.18}$$

and a gauge-dependent wave function renormalization

$$Z_2 = 1 + \frac{\alpha}{2\pi}\frac{1}{c}\frac{1}{d-4}. \tag{4.19}$$

Although the methods employed differ considerably, this is identical to positive curvature, spherical space-time results by Drummond and Shore [DS79].

[6] see Le Bellac [Bel91] 12.3.32, for example:

$$\Sigma(p) = \frac{e^2}{8\pi^2}\left[3m_e - (p\!\!\!/ - m_e)\right]\frac{1}{4-d} + \mathcal{O}((4-d)^0)$$

in the $c = 1$ Feynman gauge.

4.3 Electron-photon vertex

Consider the vertex diagram in the figure 4.4 below. This diagram could be

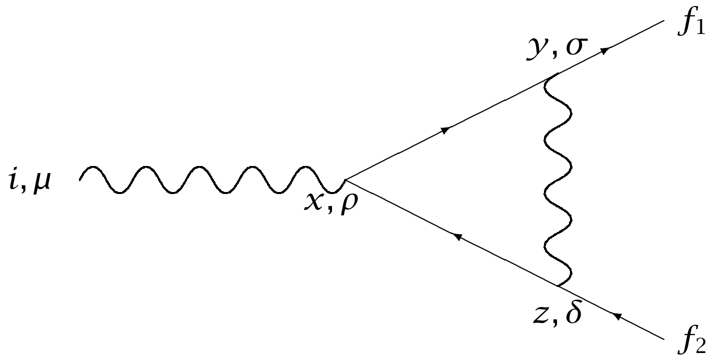


figure 4.4: The electron-photon vertex diagram.

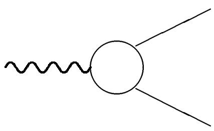
evaluated in a way completely analogous to the previous two sections. There appears to be little point to this, however. As in Minkowski space, Ward and Ward-Takahashi identities can be formulated which relate it to the electron self-energy diagram of figure 4.3. A brief derivation will be given below.

Formally, one can write the propagator of the quantized fermion field as

$$S_F(y;x) = i\langle 0|T\psi(y)\bar{\psi}(x)|0\rangle$$

for a suitable choice of the vacuum state $|0\rangle$ [Dav96]. In curved space-time the meaning of this expression is not a priori clear however. To define the time ordering operator T one needs to choose a spacelike foliation of $\mathcal{N}$. In the case of anti-de Sitter space-time, using the "Cartesian" coordinate system of (2.3), the surfaces $\{t = \text{constant}\}$ cover the space completely (even if the information coming in from timelike infinity means they are not Cauchy; this is fortunately not an issue in this context).

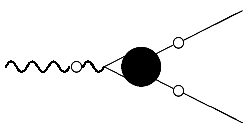
The proof is largely analogous to the textbook derivation. The vertex function



$$\begin{aligned} &= \langle 0|TA_\mu(i)\psi(f_1)\bar{\psi}(f_2)|0\rangle \\ &= ie\int_{\mathcal{N}} \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z[-g(x)]^{\frac{1}{2}}[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}} \\ &\qquad \times G_{\mu\rho}(x;i)S_F(f_2;z)\Lambda^\rho(x;y;z)S_F(y;f_1) \end{aligned}$$

can be expressed as a proper vertex with dressed external propagators

=



$$\begin{aligned} &= -i\int_{\mathcal{N}} \mathrm{d}^4x[-g(x)]^{\frac{1}{2}}G^{[0]}_{\mu\rho}(x;i) \\ &\qquad \times \langle 0|Tj^\rho(x)\psi(f_1)\bar{\psi}(f_2)|0\rangle \end{aligned}$$

The next step is to take the covariant derivative $\mathcal{D}^\mu$ with respect to the point i of both expressions. From the transversality of vector corrections and the propagator structure (3.58) and (3.65), it follows that

$$\mathcal{D}_i^\mu G_{\mu\rho}(i;x) = \mathcal{D}_i^\mu G_{\mu\rho}^{\text{long }[0]}(i;x) = \mathcal{D}_{x\rho}\mathcal{D}_i^2\kappa(x;i) = \partial_{x\rho}\mathcal{D}_i^2\kappa(x;i)$$

where $\kappa(x;i)$ is a scalar function of $\lambda(x;i)$. Partial integration gives

$$\begin{aligned} -ie\int_{\mathcal{N}} \mathrm{d}^4y\mathrm{d}^4z[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}S_F(f_2;z)\mathcal{D}_{x\rho}\Lambda^\rho(x;y;z)S_F(y;f_1) \\ = \quad i\mathcal{D}_{x\rho}\langle 0|Tj^\rho(x)\psi(f_1)\bar{\psi}(f_2)|0\rangle. \end{aligned} \tag{4.20}$$

The derivative of the volume element gave precisely the contribution to turn $\partial_\rho\Lambda^\rho$ into a proper covariant divergence — although in anti-de Sitter space, $g = -1$ and $\Gamma^\rho_{\sigma\rho}$ vanishes anyway. To avoid unnecessary clutter, the remainder of this derivation will assume anti-de Sitter space in the coordinate system (2.3) even though it can be extended to more general cases.

Given the definition of T above and the fact that j^ρ is the conserved electromagnetic current satisfying $[j_0(x),\psi(y)]\delta(x^0-y^0) = -e\psi(x)\delta^4(x-y)$, the right hand side of (4.20) works out to be

$$= \quad eS_F(f_1;f_2)\Big(\delta^4(x-f_1) - \delta^4(x-f_2)\Big)$$

Define the inverse of the spinor propagator as follows:

$$S_F^{-1}(y;x) : \int_{\mathcal{N}} \mathrm{d}^4x\; S_F^{-1}(y';x)S_F(x;y) = \delta^4(y-y'). \tag{4.21}$$

It is straightforward to show that, expressed as an operator acting on S_F,

$$\hat{S}_F^{-1}(y;x) = \Big[i\not{\mathcal{D}}_y - m - \hat{\Sigma}(y;x)\Big] \tag{4.22}$$

where $\hat{\Sigma}(y;x)$ is the electron self-energy, the first order of which has been evaluated in section 4.2. Then, left-multiplying both sides of the equation by $\int_{\mathcal{N}}\mathrm{d}^4f_2\; S_F^{-1}(z';f_2)$ and right-multiplying by $\int_{\mathcal{N}}\mathrm{d}^4f_1\; S_F^{-1}(f_1;y')$, one obtains the precise equivalent of the flat space Ward-Takahashi identity for the vertex:

$$-i\mathcal{D}_{x\rho}\Lambda^\rho(x;y;z) = S_F^{-1}(z;y)\Big(\delta^4(y-x) - \delta^4(z-x)\Big). \tag{4.23}$$

This can finally be re-expressed as an operator equation

$$-i\mathcal{D}_{x\rho}\hat{\Lambda}^\rho(x;y;z) = \Big(\delta^4(y-x) - \delta^4(z-x)\Big)\Big[i\not{\mathcal{D}}_z - m - \hat{\Sigma}(z;y)\Big] \tag{4.24}$$

relating the truncated vertex diagram $\hat{\Lambda}$ to the electron self-energy $\hat{\Sigma}$.

To recover the Ward identity, multiply both sides of the Ward-Takahashi identity with $-\int_{\mathcal{N}} d^4x\, x^\sigma$ and partially integrate:

$$\int_{\mathcal{N}} d^4x\, \Lambda^\sigma(x;y;z) = i(z-y)^\sigma S_F^{-1}(z;y) \tag{4.25}$$

or, equivalently,

$$\int_{\mathcal{N}} d^4x\, \hat{\Lambda}^\sigma(x;y;z) = i(z-y)^\sigma \left[i\not{D}_z - m - \hat{\Sigma}(z;y) \right] \tag{4.26}$$

Although it may look unfamiliar in its coordinate space guise, this is the direct analogue of the flat space Ward identity.

From this it is straightforward to argue that the renormalization constant for the first order vertex correction must equal that of the first order electron self-energy. Consider

$$\begin{aligned} \hat{\Gamma}_\mu(x;y;z) &= (Z_2^{-1}-1)\gamma_\mu - Z_2^{-1}\delta^4(y-x)i(z-y)_\mu \hat{\Sigma}^R(y;z) \\ &\quad + \hat{\Gamma}_\mu(x;y;z) - \delta^4(y-x)\int_{\mathcal{N}} d^4x'\, \hat{\Gamma}_\mu(x';y;z) \\ &\stackrel{\text{def}}{=} (Z_1^{-1}-1)\gamma_\mu + Z_1^{-1}\hat{\Gamma}_\mu^R(x;y;z) \end{aligned}$$

When y and z are taken to be connected to on mass shell propagators so that $\Sigma^R(y;z) \stackrel{\text{eff}}{=} 0$, and x is integrated over the whole of $\mathcal{N}$ (zero momentum transfer), it becomes clear that in anti-de Sitter space, as in flat space, $Z_1 = Z_2$.

chapter 5

Conclusions

In this thesis, a beginning has been made with the renormalization of quantum electrodynamics (QED) in an anti-de Sitter background space-time. This is the next step in a series of research objects which began with the study of quantized scalar fields and ϕ^4 theory in this space [FD91]. QED adds new elements: it deals with spinor and massless vector fields in the context of an Abelian field theory without anomalies. To this end, the propagators for spin 0, $\frac{1}{2}$ and 1 particles have been generalized to d-dimensional anti-de Sitter space, and the infinite parts of the vacuum polarization, self-energy and vertex processes evaluated. They turn out to be direct curved space generalizations of the familiar flat space expressions. The rescaling factors Z_1, Z_2, Z_3 and δm_e are curvature independent in the first order. The theory is thereby renormalizable at the one-loop level.

This can be said to simply confirm what you would naively expect. The ultraviolet infinities that beset quantum field theory are due to the singular short-distance behavior of the Feynman propagator. At these vanishingly small scales, you would not expect curvature to affect the physics. Apart from the confirmation of this intuitive argument by a direct computation, what has been achieved?

The answer is that the processes evaluated in this thesis represent the necessary first steps on a road that will lead to some interesting results. It is only with higher order diagrams, as divergent subdiagrams intertwine and processes occupying a finite 4-volume start to contribute, that curvature creeps into the counterterms, as borne out by the Filthaut and Dullemond paper on ϕ^4 renormalization quoted above: the *only* diagrams up to second

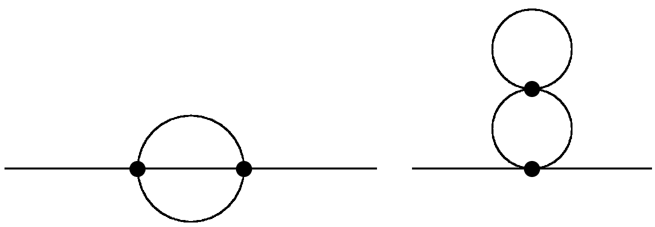


figure 5.1: The first curvature-dependent divergences in ϕ^4 theory.

order which lead to curvature dependent counterterms are precisely those of figure 5.1[1]. Renormalization of anti-de Sitter physics to higher orders will therefore not be as close to its flat space counterpart as the elementary

[1]This is understood to include their accompanying counterterms from one-loop renormalization. Regarding the results quoted in this paper it may be useful to point out that, if m_0 is the physical mass *defined* to give a conformal theory for $m_0 = 0$, then $m_0^2 = (m+1)(m+2)/R^2$. From this point of view only the diagrams of figure 5.1 exhibit any curvature dependence in their singularities.

one-loop QED processes presented here seem to suggest.

Furthermore, some of the intricacies of the programme have been explored. Generalized propagators have been calculated explicitly. A way of dealing with the discontinuous nature of the Inönü-Wigner contraction of 4-dimensional anti-de Sitter electrodynamics to Minkowski space has been proposed and successfully used to calculate the first order counterterms. The Ward-Takahashi identity for the adS vertex diagram has been employed using configuration space methods; others can be derived in similar ways. The possibilities and limitations of configuration space calculations in this theory have become apparent.

The last point touches upon one of the most important areas where this work can be progressed. It would be extremely useful to evaluate the finite parts of these diagrams and so calculate the anomalous magnetic moment of the electron, which is expected to depend on the (constant) curvature of this space. In its turn, this exercise can be used as a stepping stone for the further exploration of the interplay between the four fundamental forces of nature. It is regrettable that this thesis adds only a detail to it.

Recent years have seen interesting work with coherent states and transform-space methods in anti-de Sitter space. While there is no anti-de Sitter analog of the simple Fourier transform of momentum space, it is still possible to expand fields into modes which preserve some of the properties which make momentum space so attractive. For example, Davis [Dav96] analytically continues to Euclidean anti-de Sitter space $H^4 = \mathrm{SO}(4,1)/\mathrm{SO}(4)$ and expands into generalized plane waves: $f(\xi,\lambda) = \int dx f(x)\ (x\cdot\xi)^{-i\lambda-\frac{3}{2}}$ where ξ defines a horocycle on H^4. The massive scalar propagator, for example, can then be written as $R^2/(\lambda^2 - m^2R^2 + \frac{9}{4})$, showing that λ is analogous to the norm of the momentum. The true usefulness of this expansion lies in the fact that products of these plane waves can be simplified iteratively. The vertex factors $\prod f_{\{\sigma\}\{K\}}$ mentioned in the introduction to chapter 4 can be reduced to products of delta functions and the transform's spectral function. This technique might well be powerful enough to calculate the diagrams' finite parts.

Another expansion of possible interest is that in (Perelomov) coherent states, usually used in the context of geometric (phase space) quantization [Gaz90, GB94, AAGM95, SS99], but also in the context of propagator construction [Pol98]. These states generalize momentum states and contract to plane waves as the curvature tends to zero. It would be interesting to verify how their elegant mathematical properties carry over to the evaluation of Feynman diagrams.

Finally, even when the finite parts have been evaluated only the bare surface of both renormalization and the physics of the Standard Model will have been scratched. All the ingredients needed to prove the renormalizability of QED in anti-de Sitter space to all orders, in complete analogy to the flat Minkowski space case, are already in evidence; lack of time has unfortunately prevented the author from working this out in enough detail for incorporation in this thesis. Higher order processes can be evalated. Furthermore, it may well be feasible to prove rigorously that the whole of Standard Model physics in anti-de Sitter space is renormalizable and to calculate gravitational corrections to physical observables. Experimental verification would probably not be straightforward. But even if only a single prediction will be measurable in approximation, it will teach something about reality, whether correct or not.

appendix A

Propagator expressions

A.1 The scalar propagator

It is sometimes convenient to split the scalar propagator expressions computed in Section 3.1 into singular and analytical parts in λ and the dimension d for $d \to 4$. This has been done in (3.13) and (3.14) but will be expanded in a little more detail below. Starting from these expressions and the Feynman propagator (3.20),

$$\begin{aligned} G_F^{(0)}(\lambda) &= iR^{2-d}(4\pi)^{-\frac{d}{2}}\Gamma(m+d-1) \\ &\times\left\{\frac{\Gamma\left(\frac{2-d}{2}\right)}{\Gamma(m+1)}F\left(\tfrac{m+d-1}{2}, -\tfrac{m}{2}; \tfrac{d}{2}; \lambda\right)\right. \\ &\left.+ (-\lambda_F)^{-\frac{d-2}{2}}\frac{2^{d-2}\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(m+d-1)}F\left(\tfrac{m+1}{2}, -\tfrac{m+d-2}{2}; \tfrac{4-d}{2}; \lambda\right)\right\} \\ &= G_{F\ (\text{sing})}^{(0)}(\lambda) + G_{F\ (\text{ana})}^{(0)}(\lambda) \end{aligned} \tag{A.1}$$

$$\begin{aligned} G_{F\ (\text{sing})}^{(0)}(\lambda) &= iR^{2-d}(4\pi)^{-\frac{d}{2}} \\ &\times\left\{(-\lambda_F)^{-\frac{d-2}{2}}2^{d-2}\Gamma\left(\tfrac{d-2}{2}\right)F\left(\tfrac{m+1}{2}, -\tfrac{m+d-2}{2}; \tfrac{4-d}{2}; \lambda\right)\right. \\ &\left.- \frac{2\mu^{d-4}}{4-d}(4\pi R^2)^{\frac{d-4}{2}}(m+1)(m+2)F\left(\tfrac{m+3}{2}, -\tfrac{m}{2}; 2; \lambda\right)\right\} \\ &= \frac{1}{4\pi^2 iR^2\lambda_F}\left\{\frac{\Gamma\left(\frac{d-2}{2}\right)}{(-\pi R^2\lambda_F)^{\frac{d-4}{2}}} + \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\left(\frac{m+1}{2}\right)_k\right. \\ &\left.\times\left[\frac{(-\frac{m+d-2}{2})_k\Gamma\left(\frac{d-2}{2}\right)}{(\frac{4-d}{2})_k(-\pi R^2\lambda)^{\frac{d-4}{2}}} - \frac{(-\frac{m+2}{2})_k}{\Gamma(k)}\frac{2\mu^{d-4}}{4-d}\right]\right\} \end{aligned} \tag{A.2}$$

$$\begin{aligned} G_{F\ (\text{ana})}^{(0)}(\lambda) &= iR^{2-d}(4\pi)^{-\frac{d}{2}} \\ &\times\left\{\frac{\Gamma\left(\frac{2-d}{2}\right)\Gamma(m+d-1)}{\Gamma(m+1)}F\left(\tfrac{m+d-1}{2}, -\tfrac{m}{2}; \tfrac{d}{2}; \lambda\right)\right. \\ &\left.+ \frac{2\mu^{d-4}}{4-d}(4\pi R^2)^{\frac{d-4}{2}}(m+1)(m+2)F\left(\tfrac{m+3}{2}, -\tfrac{m}{2}; 2; \lambda\right)\right\} \\ &= \frac{1}{16\pi^2 iR^2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\left(-\frac{m}{2}\right)_k\left[\frac{\Gamma(m+d-1)}{\Gamma(m+1)}\frac{(\frac{m+d-1}{2})_k\Gamma\left(\frac{2-d}{2}\right)}{(\frac{d}{2})_k(4\pi R^2)^{\frac{d-4}{2}}}\right. \\ &\left.+ (m+1)(m+2)\frac{(\frac{m+3}{2})_k}{\Gamma(k+2)}\frac{2\mu^{d-4}}{4-d}\right] \end{aligned} \tag{A.3}$$

where the length scale μ has been introduced to get the dimensions right. The $d \to 4$ pole terms in $G_{F\ (\text{ana})}$ have been chosen so that they cancel precisely. $G_{F\ (\mathrm{sing})}$ exhibits both $\lambda \to 0$ and $d \to 4$ singularities, but in different terms — as noted in the main text, this is a necessary property for the renormalization procedure to work.

Since the derivative of G_F plays a role in computations vacuum polarization and electron self-energy, it is useful to list the corresponding expressions for G'_F here:

$$
\begin{aligned}
G_F^{'(0)}(\lambda) &= i\frac{R^{2-d}}{4(4\pi)^{\frac{d}{2}}} \\
&\times\left\{\frac{\Gamma(m+d)\,\Gamma\left(-\frac{d}{2}\right)}{\Gamma(m)}F\left(\tfrac{m+d+1}{2},-\tfrac{m-2}{2};\tfrac{d+2}{2};\lambda\right)\right. \\
&\quad+(-\lambda_F)^{-\frac{d}{2}}2^d\Gamma\left(\tfrac{d}{2}\right)F\left(\tfrac{m+1}{2},-\tfrac{m+d-2}{2};\tfrac{4-d}{2};\lambda\right) \\
&\quad+(-\lambda_F)^{-\frac{d-2}{2}}2^{d-2}(m+1)(m+d-2)\Gamma\left(\tfrac{d-4}{2}\right) \\
&\quad\left.\times F\left(\tfrac{m+3}{2},-\tfrac{m+d-4}{2};\tfrac{6-d}{2};\lambda\right)\right\} \qquad \text{(A.4)} \\
&= G_{F\ (\mathrm{sing})}^{'(0)}(\lambda)+G_{F\ (\mathrm{ana})}^{'(0)}(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
G_{F\ (\mathrm{sing})}^{'(0)}(\lambda) &= i\frac{R^{2-d}}{4(4\pi)^{\frac{d}{2}}}\left\{\mu^{d-4}(4\pi R^2)^{\frac{d-4}{2}}\frac{m(m+1)(m+2)(m+3)}{4-d}\right. \\
&\quad\times F\left(\tfrac{m+5}{2},-\tfrac{m-2}{2};3;\lambda\right) \\
&\quad+(-\lambda_F)^{-\frac{d}{2}}2^d\Gamma\left(\tfrac{d}{2}\right)F\frac{m+1}{2}-\frac{m+d-2}{2}\frac{4-d}{2}\lambda \\
&\quad+(-\lambda_F)^{-\frac{d-2}{2}}2^{d-2}(m+1)(m+d-2)\Gamma\left(\tfrac{d-4}{2}\right) \\
&\quad\left.\times F\left(\tfrac{m+3}{2},-\tfrac{m+d-4}{2};\tfrac{6-d}{2};\lambda\right)\right\} \\
&= -\frac{1}{4\pi^2 iR^2\lambda_F^2}\left\{\Gamma\left(\tfrac{d}{2}\right)(-\pi R^2\lambda_F)^{\frac{4-d}{2}}\right. \\
&\quad+\lambda_F\frac{(m+1)(m+d-2)}{4}\Gamma\left(\tfrac{d-2}{2}\right)(-\pi R^2\lambda_F)^{\frac{4-d}{2}} \\
&\quad+\sum_{k=2}^{\infty}\frac{\lambda^k}{k!}\left(\frac{m+1}{2}\right)_k\left[\frac{(-\frac{m+d-2}{2})_k\Gamma\left(\frac{d}{2}\right)}{(\frac{2-d}{2})_k(-\pi R^2\lambda)^{\frac{d-4}{2}}}\right. \\
&\quad\left.\left.+\frac{(-\frac{m+2}{2})_k}{\Gamma(k-1)}\frac{2\mu^{d-4}}{4-d}\right]\right\} \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
G_{F\ (\mathrm{ana})}^{'(0)}(\lambda) &= i\frac{R^{2-d}}{4(4\pi)^{\frac{d}{2}}}\left\{\frac{\Gamma(m+d)\,\Gamma\left(-\frac{d}{2}\right)}{\Gamma(m)}\right. \\
&\quad\times F\left(\tfrac{m+d+1}{2},-\tfrac{m-2}{2};\tfrac{d+2}{2};\lambda\right)
\end{aligned}
$$

$$
\begin{aligned}
& - \mu^{d-4} (4\pi R^2)^{\frac{d-4}{2}} \frac{m(m+1)(m+2)(m+3)}{4-d} \\
& \quad \times F\left(\tfrac{m+5}{2}, -\tfrac{m-2}{2}; 3; \lambda\right)\Big\} \\
= \; & -\frac{1}{64\pi^2 i R^2} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (-\frac{m-2}{2})_k \\
& \times \Bigg[\frac{\Gamma(m+d)\,\Gamma\left(-\frac{d}{2}\right)}{\Gamma(m)} \frac{(\frac{m+d+1}{2})_k}{(\frac{d+2}{2})_k (4\pi R^2)^{\frac{d-4}{2}}} \\
& \quad - m(m+1)(m+2)(m+3) \frac{(\frac{m+5}{2})_k}{(3)_k} \frac{\mu^{d-4}}{4-d} \Bigg]
\end{aligned} \tag{A.6}
$$

The observations made for G_F also hold true for its derivative.

A.2 The vector propagator

Expressions for the vector propagator will be worked out in the following with a bit more detail than the main text provides. Analytically continuing the longitudinal solution (3.64) to the $|\lambda| < 1$ region, one finds for fractional d

$$
\begin{aligned}
\kappa(\lambda) = \; & \Gamma\left(m' + \tfrac{d+1}{2}\right) \Bigg[\frac{\Gamma\left(-\frac{d-2}{2}\right)}{\Gamma\left(\frac{m'+1}{2}\right)\Gamma\left(\frac{m'+2}{2}\right)} F\left(\tfrac{m'+d-1}{2}, -\tfrac{m'}{2}; \tfrac{d}{2}; \lambda\right) \\
& + \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{m'+d-1}{2}\right)\Gamma\left(\frac{m'+d}{2}\right)} (-\lambda)^{-\frac{d-2}{2}} F\left(\tfrac{m'+1}{2}, -\tfrac{m'+d-2}{2}; -\tfrac{d-4}{2}; \lambda\right) \Bigg]
\end{aligned} \tag{A.7}
$$

Of course, not $\kappa(\lambda)$ itself is needed, but the first and second derivative with respect to z (see eqn. 3.58). This is fairly straightforward; using [AS84] 15.3.3 and (2.36) ($\lambda = 1 - z^2$, so $\frac{d}{dz} = -2z\frac{d}{d\lambda}$):

$$
\begin{aligned}
\kappa'(\lambda) = \; & -2\Gamma\left(m' + \tfrac{d+1}{2}\right) \Bigg[\frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(\frac{m'}{2}\right)\Gamma\left(\frac{m'+1}{2}\right)} \\
& \times \frac{m'+d-1}{2} F\left(\tfrac{m'+d}{2}, -\tfrac{m'-1}{2}; \tfrac{d+2}{2}; \lambda\right) \\
& + \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)} (-\lambda)^{-\frac{d}{2}} F\left(\tfrac{m'}{2}, -\tfrac{m'+d-1}{2}; -\tfrac{d-2}{2}; \lambda\right) \Bigg]
\end{aligned} \tag{A.8}
$$

$$\kappa''(\lambda) = 4\Gamma\left(m' + \tfrac{d+1}{2}\right)\Bigg[\frac{\Gamma\left(-\frac{d+2}{2}\right)}{\Gamma\left(\frac{m'-1}{2}\right)\Gamma\left(\frac{m'}{2}\right)}$$
$$\times \frac{m'+d-1}{2}\,\frac{m'+d}{2}F\left(\tfrac{m'+d+1}{2}, -\tfrac{m'-2}{2}; \tfrac{d+4}{2}; \lambda\right)$$
$$+\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}(-\lambda)^{-\frac{d+2}{2}}F\left(\tfrac{m'-1}{2}, -\tfrac{m'+d}{2}; -\tfrac{d}{2}; \lambda\right)\Bigg] \tag{A.9}$$

The transversal solution's $f(\lambda)$ and $g(\lambda)$ functions (3.72) and (3.73), continued to the $|\lambda| < 1$ region, are

$$f(\lambda) = \Gamma\left(m + \tfrac{d+1}{2}\right)$$
$$\times\Bigg[\frac{\Gamma\left(\frac{d-2}{2}\right)(-\lambda)^{-\frac{d}{2}}}{\Gamma\left(\frac{m+d-1}{2}\right)\Gamma\left(\frac{m+d}{2}\right)}\Big\{(d-2)F\left(\tfrac{m-1}{2}, -\tfrac{m+d-2}{2}; -\tfrac{d-2}{2}; \lambda\right)$$
$$+ m(m+d-2)(-\lambda)F\left(\tfrac{m+1}{2}, -\tfrac{m+d-2}{2}; -\tfrac{d-4}{2}; \lambda\right)\Big\}$$
$$+\frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}\Big\{(m-1)F\left(-\tfrac{m-2}{2}, \tfrac{m+d-1}{2}; \tfrac{d+2}{2}; \lambda\right)$$
$$- d(m+d-2)F\left(-\tfrac{m}{2}, \tfrac{m+d-1}{2}; \tfrac{d}{2}; \lambda\right)\Big\}\Bigg] \tag{A.10}$$

$$g(\lambda) = \Gamma\left(m + \tfrac{d+1}{2}\right)$$
$$\times\Bigg[\frac{\Gamma\left(\frac{d}{2}\right)(-\lambda)^{-\frac{d+2}{2}}}{\Gamma\left(\frac{m+d-1}{2}\right)\Gamma\left(\frac{m+d}{2}\right)}\Big\{-2dF\left(\tfrac{m-2}{2}, -\tfrac{m+d-1}{2}; -\tfrac{d}{2}; \lambda\right)$$
$$+ 2(m+d-2)(-\lambda)F\left(\tfrac{m}{2}, -\tfrac{m+d-1}{2}; -\tfrac{d-2}{2}; \lambda\right)\Big\}$$
$$-\frac{\Gamma\left(-\frac{d+2}{2}\right)(m+d-1)}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}\Big\{\frac{(m-1)(m-2)}{2}F\left(-\tfrac{m-3}{2}, \tfrac{m+d}{2}; \tfrac{d+4}{2}; \lambda\right)$$
$$+ (m+d-2)\frac{d+2}{2}F\left(-\tfrac{m-1}{2}, \tfrac{m+d}{2}; \tfrac{d+2}{2}; \lambda\right)\Big\}\Bigg] \tag{A.11}$$

The linear combination canceling the singularities as found in section 3.3 (3.76) is, expanded into a power series in λ,

$$\begin{aligned}\mathcal{F}(\lambda) &= \frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(m+\frac{d+1}{2}\right)}f(\lambda) + \frac{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}{\Gamma\left(m'+\frac{d+1}{2}\right)}\kappa'(\lambda)\\ &= \Gamma\left(\tfrac{d-2}{2}\right)\frac{1+c}{2c}\,\frac{(m+1)(m+d-2)}{(-\lambda)^{\frac{d-2}{2}}}\end{aligned}$$

$$-\frac{1}{16}\Gamma\left(\tfrac{d-4}{2}\right)\Bigg[(m+1)(m+d-2)\frac{1+3c^2}{c^2} + 2(d-1)\frac{1-c}{c} - \frac{4}{c}\Bigg]\frac{(m+1)(m+d-2)}{(-\lambda)^{\frac{d-4}{2}}} - \frac{1}{2^{d-1}}\Gamma\left(-\tfrac{d}{2}\right)\Bigg[\frac{\Gamma(m+d)}{\Gamma(m)}(d-1) + \frac{\Gamma(m'+d)}{\Gamma(m')}\Bigg] + \dots \tag{A.12}$$

up to $\mathcal{O}(\lambda^0)$. The same linear combination for the $\mathcal{G}$ part yields, up to $\mathcal{O}(\lambda^{-1})$,

$$\begin{aligned}\mathcal{G}(\lambda) &= \frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(\frac{m+d-1}{2}\right)}{\Gamma\left(m+\frac{d+1}{2}\right)}g(\lambda) + \frac{\Gamma\left(\frac{m'+d}{2}\right)\Gamma\left(\frac{m'+d-1}{2}\right)}{\Gamma\left(m'+\frac{d+1}{2}\right)}\kappa''(\lambda) \\ &= \Gamma\left(\tfrac{d}{2}\right)\frac{c-1}{c}\frac{(m+1)(m+d-2)}{(-\lambda)^{\frac{d}{2}}} \\ &\quad -\frac{1}{8}\Gamma\left(\tfrac{d-2}{2}\right)\left\{\frac{c^2-1}{c^2}\frac{(m+1)^2(m+d-2)^2}{(-\lambda)^{\frac{d-2}{2}}} + 2\frac{c+1}{c}\frac{(m+1)(m+d-2)}{(-\lambda)^{\frac{d-2}{2}}}\right\} + \dots \end{aligned} \tag{A.13}$$

so that, up to the same order, the full, normalized Feynman vector propagator on the principal sheet works out to be

$$\begin{aligned} K_F^{(0)\;MN}(y;x) = &-\frac{\Gamma\left(\frac{d-2}{2}\right)}{8i\pi^{\frac{d}{2}}R^{d-2}}\Bigg\{P^M(y)\cdot P^N(x)\Bigg[\frac{1+c}{c}(-\lambda_F)^{-\frac{d-2}{2}} \\ &- \frac{1}{4(d-4)}\left\{(m+1)(m+d-2)\frac{1+3c^2}{c^2} + 2(d-1)\frac{1-c}{c} - \frac{4}{c}\right\}(-\lambda_F)^{-\frac{d-4}{2}} \\ &- \frac{\Gamma\left(-\frac{d}{2}\right)}{2^{d-2}\Gamma\left(\frac{d-2}{2}\right)}\left\{\frac{\Gamma(m+d-2)}{\Gamma(m+2)}m(m+d-1)(d-1) + \frac{\Gamma(m'+d-1)}{c\Gamma(m'+1)}\right\} + \mathcal{O}(\lambda^{\frac{6-d}{2}})\Bigg] \\ &+ P^M(y)\cdot\hat{x}\;\hat{y}\cdot P^N(x)\Bigg[(d-2)\frac{c-1}{c}(-\lambda_F)^{-\frac{d}{2}} \\ &- \frac{1}{4}\left\{\frac{c^2-1}{c^2}(m+1)(m+d-2) + 2\frac{c+1}{c}\right\}(-\lambda_F)^{-\frac{d-2}{2}} \\ &+ \mathcal{O}(\lambda^{\frac{4-d}{2}})\Bigg]\Bigg\}. \end{aligned} \tag{A.14}$$

As was the case for the scalar propagator, the $\lambda \to 0$ and $d \to 4$ singularities nowhere appear in one and the same term. In fact the $1/(d-4)$ poles in the third through the sixth line cancel each other in the $d \to 4$ limit for arbitrary m, giving rise to a $\ln(-\lambda)$ type term in complete analogy with the scalar propagator (3.12):

$$
\begin{aligned}
K^{(0)\,MN}_{F,d=4}(y;x) = -\frac{1}{8i\pi^2R^2}\Big\{&P^M(y)\cdot P^N(x)\Big[\frac{1+c}{c}(-\lambda_F)^{-1}\\
&+\frac{1-3c}{4c}\ln(-\lambda/4)+(m+1)(m+2)\frac{1+3c^2}{8c^2}\ln(-\lambda/4)\\
&+\frac{1}{8}\Big\{-2(m+1)(m+5)+5-\frac{4}{c}(\psi(m'+1)-\psi(1))\\
&\quad-2\frac{(m+1)(m+2)}{c^2}\psi(m'+1)-\frac{4m'+7}{c}\\
&\quad-2(m+1)\frac{1+3c^2}{c^2}\Big(1-(m+2)(\frac{3}{4}-\gamma)\Big)\\
&\quad-6(m(m+3)\psi(m+2)+2\psi(1))\Big\}+\mathcal{O}(\lambda_F\ln(-\lambda))\Big]\\
+P^M(y)\cdot\hat{x}\;\hat{y}\cdot P^N(x)\Big[&-2\frac{1-c}{c}(-\lambda_F)^{-2}\\
&-\frac{1}{4}\Big\{-\frac{1-c^2}{c^2}(m+1)(m+2)+2\frac{1+c}{c}\Big\}(-\lambda_F)^{-1}\\
&+\mathcal{O}(\ln(-\lambda))\Big]\Big\}.
\end{aligned}
\tag{A.15}
$$

For the gauge choice $c = 1/3$, the massless 4-dimensional vector propagator gets a particularly simple form where the logarithmic term disappears completely. Direct substitution in the expressions for $\mathcal{F}$ and $\mathcal{G}$ shows this to be true to all orders.

Another important observation is that although normalization of K_F has divided the preceding expressions by the mass squared, none of its terms become singular for the massless ($m = -1, m' = 0$) case. That this is true to all orders in λ can be proven by simple substitution in the full expressions.

appendix B

Dimensional regularization

B.1 Minkowski space

Performing QFT calculations in configuration space one often encounters expressions of the type

$$\int \mathrm{d}^4 z \frac{f(z)}{(z^2)^m}$$

where $m > 2$, i.e. the argument has a pole for $z \to 0$ and the integral diverges. When evaluating localized physical processes it can be safely assumed that $f(z)$ falls off rapidly for $|z| \to \infty$. To make sense of the physics, these expressions are first made finite (regularization) and the terms that constitute an infinity absorbed into the parameters of the theory (renormalization).

One popular way of regularizing the above expression is moving to a fractional number of dimensions [tHV72]:

$$\int \mathrm{d}^d z \frac{f(z)}{(z^2)^m}$$

upon which the integral becomes finite for most choices of d. The singularity will crop up as a $d \to 4$ pole and can be removed by renormalizing the theory's parameters with a $d \to 4$ pole counterterm.

Isolating the singular terms from integrals like the one quoted above is therefore the first step towards renormalization. The way of doing this is basically straightforward. First make a Taylor expansion of $f(z)$; only the even terms need to be considered, since the odd ones vanish in the integration:

$$f(z) \overset{\text{eff}}{=} \sum_l a_l (z^2)^l$$

Then Wick rotate the integration path to create a Euclidean integral.

$$\begin{aligned} \text{Pole} \int \mathrm{d}^d z \frac{f(z)}{(z^2)^m} &= \sum_l a_l \ \text{Pole} \int \mathrm{d}^d z (z^2)^{l-m} \\ &= \ -i \sum_l a_l (-)^{l-m} \ \text{Pole} \int \mathrm{d}^{d-1}\Omega \int_0^\infty \mathrm{d}r r^{2(l-m)+d-1} \end{aligned}$$

The angular part can be directly computed, so if $\Omega_d \overset{\text{def}}{=} \int \mathrm{d}^{d-1}\Omega = 2\pi^{d/2}\Gamma(\frac{d}{2})^{-1}$ is the total angle in d dimensions,

$$\begin{aligned} &= \ -i\Omega_d \sum_l a_l (-)^{l-m} \ \text{Pole} \int_0^\infty \mathrm{d}r r^{2(l-m)+d-1} \\ &= \ -i\Omega_d \sum_l a_l (-)^{l-m} \frac{\mu^{d-2m+2l}}{d-2m+2l} \end{aligned}$$

where the r integral has been evaluated by integrating $r^{2l-2m+d-1}$, analytically continuing to $d = 2(m-l)+\varepsilon$, evaluating the $r \sim 0$ region of the integral ($r \to \infty$ does not contribute based on the assumptions made for the asymptotic behavior of $f(z)$), then analytically continuing back. Provided there are no cuts in the complex d plane, this procedure is well defined. The length scale μ has to be introduced in order to preserve dimensional correctness. Note that for a given dimension D only *one* term in the Taylor expansion contributes to the $d \to D$ pole: assuming that m is possibly a function $m(d)$ of the dimension d,

$$\text{Pole}\lim_{d\to D}\int \mathrm{d}^d z \frac{f(z)}{(z^2)^{m(d)}} = -i(-)^{-D/2}\Omega_D \frac{a_{m(D)-D/2}}{1-2m'(D)}\,\frac{\mu^{d-D}}{d-D}. \tag{B.1}$$

At the first encounter you may, like the author, have found this way of evaluating integrals questionable. The same result can be achieved by other means however, provided the function $f(z)$ is sufficiently well-behaved for $z \to \infty$. The parallels with the anti-de Sitter problem discussed in the next section are sufficiently close for it to merit some attention. Consider the following expansion of (the even part of) f:

$$f(z) \stackrel{\text{eff}}{=} \sum_l b_l\,(z^2)^l e^{z^2/\mu^2}$$

The a_l coefficients can be expressed in terms of b_l:

$$a_m = \sum_{l=0}^{m} \frac{1}{\mu^{2(m-l)}(m-l)!} b_l$$

and a recursion relation will yield b_l in terms of a_l. The nice thing about this expansion is that the integrals can be computed directly now:

$$\begin{aligned}
\int \mathrm{d}^d z\,\frac{f(z)}{(z^2)^m} &= \sum_l b_l \int \mathrm{d}^d z\; e^{z^2/\mu^2}(z^2)^{l-m} \\
&= -i\Omega_d \sum_l b_l (-)^{l-m}\int \mathrm{d}r\; e^{-r^2/\mu^2} r^{-2(m-l)+d-1} \\
&= -i\frac{1}{2}\Omega_d \sum_l b_l(-)^{l-m}\mu^{d-2m+2l}\Gamma(-m+l+\frac{d}{2})
\end{aligned}$$

Since the last step involves an integral representation of Γ valid only for $d > 2(m-l)$ here, too, an analytical continuation to $d = 2(m-l)+\varepsilon$ is necessary. Analytically continuing back, the poles of the Γ function are well known:

$$\begin{aligned}&\text{Pole}\lim_{d\to D}\int \mathrm{d}^d z\frac{f(z)}{(z^2)^{m(d)}}\\ &= \quad -i(-)^{-\frac{D}{2}}\Omega_D\sum_{l=0}^{m(D)-\frac{D}{2}}\frac{\mu^{2(l-m+\frac{D}{2})}b_l}{(m(D)-l-\frac{D}{2})!}\frac{1}{1-2m'(D)}\frac{\mu^{d-D}}{d-D}\end{aligned}$$

This can be seen to be identical to eqn. (B.1), reinforcing confidence that the approach is valid, at least for smooth functions vanishing sufficiently fast at infinity. The expressions one typically encounters in physical theory satisfy these conditions.

As a matter of fact, this expansion of f allows full evaluation of the finite part to order $(d-D)^0$. Assuming that $f(x;d)$ and hence $b_l(d)$ can depend on the dimensionality as well as $m(d)$, but suppressing that dependency in both b_l and m for the sake of clarity (all are evaluated at $d = D$):

$$\begin{aligned}\lim_{d\to D}\int \mathrm{d}^d z\frac{f(z)}{(z^2)^{m(d)}} &= -i(-)^{-\frac{D}{2}}\frac{1}{2}\Omega_D\Bigg\{\sum_{l=0}^{m-\frac{D}{2}} b_l\frac{\mu^{2(-m+l+\frac{D}{2})}}{(m-l-\frac{D}{2})!}\\ &\times\frac{1}{1-2m'}\frac{\mu^{d-D}}{d-D}\Big[2+(1-2m')(d-D)\Big(\psi(m-l-\frac{D}{2})+\ln\mu\Big)\\ &\quad + (d-D)\Big(\ln\pi-\psi(\frac{D}{2})-2\pi i m'+2\frac{b_l'}{b_l}\Big)\Big]\\ &+\sum_{l=m-\frac{D}{2}+1}^{\infty} b_l(-)^{-m+l+\frac{D}{2}}\mu^{2(-m+l+\frac{D}{2})}\Gamma(-m+l+\frac{D}{2})\Bigg\}\end{aligned}$$

The function ψ is the well known digamma function.

B.2 Anti-de Sitter space

In anti-de Sitter space regularization – or, rather, its covering space – one encounters integrals of the form

$$R^{-2m}\int_{\mathcal{N}} \mathrm{d}^d x\frac{f^{(k)}(\lambda(y;x))}{\lambda(y;x)^m}$$

where y is integrated over the entire space $\mathcal{N}$, and $\lambda(y;x)$ is as defined in (2.36). In practice, it will be embedded in $\bar{\mathcal{M}} = \mathbb{E}(d-1,2)\otimes\mathbb{Z}$ (see chapters 2.4 and 2.5). Since $\lambda(y;x) = 1 - z^2(y;x)$ (eqn. 2.37) and z can, in turn, be expressed in terms of coordinates in $\mathcal{N}$(eqns. (2.40) and (2.41)), a series expansion for the above integrand can be made and the individual terms evaluated using eqn. (B.1).

Although this approach works fine, the integral can actually be evaluated almost directly. After Taylor expanding

$$f^{(k)}(\lambda(y;x)) = \sum_l R^{2l} a_l \lambda(y;x)^l$$

where $\lambda(y;x)$ represents a function of the coordinates in $\mathcal{N}$, i.e. *including* the phase factors as discussed in section 2.5, one is left with expressions of the form

$$\begin{aligned}
&\int_{\mathcal{N}} \mathrm{d}^d x\ \lambda(y;x)^{-n} \\
&= \int_{\mathcal{N}} \mathrm{d}^d x\ \lambda(0;x)^{-n} \\
&= \Omega_{d-1} \int_{-\infty}^{\infty} \mathrm{d}t \int_0^{\infty} r^{d-2}\mathrm{d}r \left(\sin^2\frac{t}{R} - \frac{r^2}{R^2}\cos^2\frac{t}{R}\right)^{-n} \\
&= R^d \Omega_{d-1} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^{\infty} \mathrm{d}\rho \frac{\rho^{d-2}}{\left(\sin^2\tau - \rho^2\cos^2\tau\right)^n}
\end{aligned}$$

where $\lambda(0;x)$ was given in eqn. (2.42). The ρ integral evaluates to a beta function (see Abramowitz and Stegun [AS84] 6.2.1):

$$\begin{aligned}
&= (-)^{-n}\frac{1}{2}R^d\Omega_{d-1}B\Big(\frac{d-1}{2}, n - \frac{d-1}{2}\Big) \\
&\quad \times \int_{-\infty}^{\infty} \mathrm{d}\tau (-\sin\tau)^{d-1-2n}(\cos\tau)^{1-d}
\end{aligned}$$

The interest is in the singular part of this expression. As it stands there are singularities for all $\tau = 2\pi m, m \in \mathbb{Z}$ but physically, it is clear that this comes from the region $x \approx y \Leftrightarrow \tau \approx 0$.

This is where attention should be paid to the actual integration path. As argued in section 3.1, the path in the neighborhood of the singularities depends crucially on the type of propagator expression evaluated. In the Feynman propagators, $\lambda(y;x)$ is replaced by $\lambda_F(y;x)$ (3.21) denoting, when run over the entire CadS covering space, a contour passing the poles in the upper $\mathbb{C}$ plane for $\tau < 0$, in the lower for $\tau > 0$, and the $\tau = 0$ singularity as shown in figure B.1. The integration path in this picture is the real axis. The

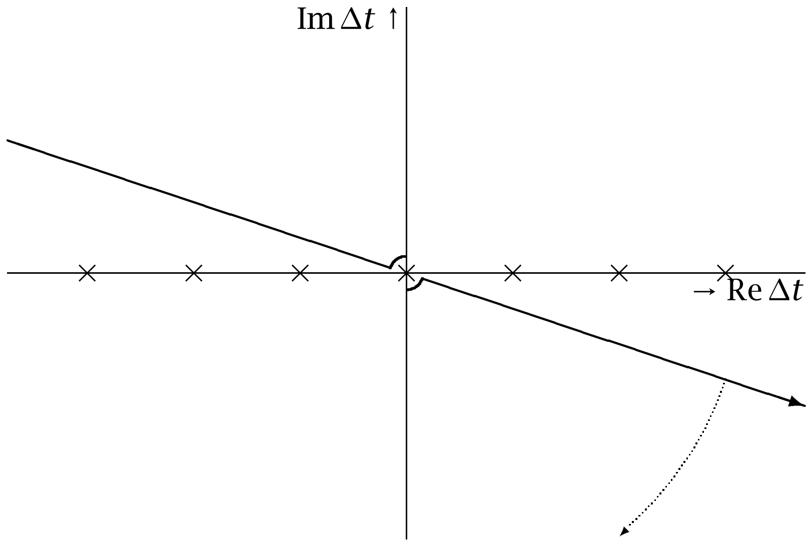


figure B.1: the integration path for Feynman propagators in anti-de Sitter space

next step is to get rid of most of the singularities by deforming this path. Given an $f(\lambda)$ with no singularities in the complex τ plane apart from $\tau = \infty$ and possibly $\tau = 0$, vanishing rapidly enough at infinity, a Wick rotation can be performed[1], i.e. the path tilted as shown by the dotted arrow in the figure:

$$\begin{aligned} = \quad & (-)^{-n}\frac{1}{2}R^d\Omega_{d-1}B\Big(\frac{d-1}{2}, n - \frac{d-1}{2}\Big) \\ & \times \int_{-\infty}^{\infty} \mathrm{d}\tau \sinh^{d-1-2n}\tau \cosh^{1-d}\tau \end{aligned}$$

The singularity in the origin ($x = y$, identical sheets in $\bar{\mathcal{M}}$) is the only one that cannot be evaded[2]. The tilted contour now describes a straight path through this singularity and can directly be computed. The only point of concern is the possible contribution from the arcs at infinity, even given a vanishing $f(\lambda)$, because of the recurring singularities around the real τ axis. However, as Filthaut [Fil89] pointed out, the same infinitesimal time modification (3.21) that was used to prescribe the integration path also introduces an exponential fall-off as $|\tau| \to \infty$.

It is worth noting that this makes the entire expression dependent on the behavior in the principal $k_x = k_y$ sheet only. Neither the infinite series

[1]In fact, the integral over ρ already presupposed this Wick rotation in its implicit assumption that $-\sin^2\tau$ could be treated as if it were positive.

[2]Or, equivalently, the path is pinched between the two singularities around the origin, if that more familiar formulation is chosen — see the discussion just before (3.20).

of singularities, representing the reflection of waves against spatial infinity, nor the phase factors that crop up when going to the covering space (2.49) have any import here.

For $\operatorname{Re} d - 2n > 0$ and $\operatorname{Re} n > 0$ the quoted integral is well defined[3], so analytical continuation to $d = 2n + \varepsilon$ gives

$$\begin{aligned} &= -i(-)^{-n} R^d \Omega_{d-1} B\Big(\frac{d-1}{2}, n - \frac{d-1}{2}\Big) B\Big(\frac{d}{2} - n, n\Big) \\ &= -i(-)^{-n} \frac{1}{2} R^d \Omega_d B\Big(n - \frac{d-1}{2}, \frac{d}{2} - n\Big) \end{aligned}$$

In the $d \to D$ limit, $\Gamma\left(\frac{d}{2} - n\right)$ exhibits $d = D$ poles at all $n \in \mathbb{N} : n >= \frac{D}{2}$. For given $n(d)$ the pole part is

$$\begin{aligned} &\operatorname{Pole} \lim_{d \to D} \int_{\mathcal{N}} \mathrm{d}^d x \lambda(y; x)^{-n(d)} \\ &= -i(-)^{-\frac{D}{2}} R^D \Omega_d \frac{\Gamma\left(n(D) - \frac{D-1}{2}\right)}{\pi^{\frac{1}{2}} \Gamma\left(n(D) - \frac{D-2}{2}\right)} \frac{1}{1 - 2n'(D)} \frac{\mu^{d-D}}{d - D} \end{aligned} \tag{B.2}$$

[3] The integral is most easily evaluated by constructing it using simple integer powers first. Repeated application of [AS84] 4.5.86 gives the following result for an integration of $\cosh^{-2l} \tau$:

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \cosh^{-2l} \tau = \frac{(2l-2)(2l-4)\cdots(2)}{(2l-1)(2l-3)\cdots(3)} \int_{-\infty}^{\infty} \mathrm{d}\tau \cosh^{-2} \tau = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(l)}{\Gamma\left(l + \frac{1}{2}\right)}$$

Using 4.5.85 it is possible to iteratively bring in a $\sinh^{2k}$ function as well:

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}\tau \sinh^{2k} \tau \cosh^{-2l} \tau \\ &= (-)^k \frac{(2k-1)(2k-3)\cdots(1)}{(2k-2l)(2k-2l-2)\cdots(2-2l)} \int_{-\infty}^{\infty} \mathrm{d}\tau \cosh^{-2l} \tau \\ &= \frac{\Gamma\left(k + \frac{1}{2}\right)\Gamma(l-k)}{\Gamma\left(l + \frac{1}{2}\right)}. \end{aligned}$$

provided $k < l$. Analytical continuation to arbitrary powers $\alpha, \beta \in \mathbb{C}$, with $\operatorname{Re} \alpha > -1, \operatorname{Re} \beta + \operatorname{Re} \alpha < 0$ then yields the beta function

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \sinh^{\alpha} \tau \cosh^{\beta} \tau = B\Big(\frac{\alpha+1}{2}, -\frac{\alpha+\beta}{2}\Big).$$

Apparently in anti-de Sitter space all orders of f up to $\mathcal{O}(\lambda^{m-2})$ contribute:

$$
\begin{aligned}
&\operatorname{Pole}\lim_{d\to D} R^{-2m(D)} \int_{\mathcal{N}} \mathrm{d}^d y \frac{f(\lambda(y;x))}{\lambda(y;x)^{m(d)}} \\
&= -i(-)^{-\frac{D}{2}} R^{D-2m(D)} \Omega_D \sum_{l=0}^{m(D)-\frac{D}{2}} R^{2l} a_l \\
&\quad \times \frac{\Gamma\left(m(D)-l-\frac{D-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(m(D)-l-\frac{D-2}{2}\right)} \frac{1}{1-2m'(D)} \frac{\mu^{(d-D)(1-2m'(D))}}{d-D}
\end{aligned}
\tag{B.3}
$$

Apart from the contribution of all $a_{l\le m(D)-D/2}$ this expression does, not unexpectedly, closely resemble the Minkowski case in the preceding section. Indeed it is easily verified that in the $R\to\infty$ limit only the $a_{m(D)-D/2}$ term survives and exactly reproduces (B.1) — something to inspire confidence in the results so far.

As was the case in Minkowski space, the finite part can be evaluated as well.

$$
\begin{aligned}
&\lim_{d\to D} R^{-2m(d)} \int_{\mathcal{N}} \mathrm{d}^d y \frac{f(\lambda(y;x))}{\lambda(y;x)^{m(d)}} \\
&= -i(-)^{-\frac{D}{2}} \frac{1}{2} R^{D-2m} \Omega_D \Bigg\{ \sum_{l=0}^{m-\frac{D}{2}} R^{2l} a_l \frac{\Gamma\left(m-l-\frac{D-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(m-l-\frac{D-2}{2}\right)} \\
&\quad \times \frac{1}{1-2m'} \frac{\mu^{(d-D)(1-2m')}}{d-D} \Big[2 + (1-2m')(d-D) \\
&\qquad \times \Big(\psi(m-l-\frac{D}{2}) - \psi(m-l-\frac{D-1}{2}) + \ln R^2\Big) \\
&\qquad + (d-D)\Big(\ln\pi - \psi(\frac{D}{2}) - 2\pi i m' + 2\frac{a_l'}{a_l}\Big)\Big] \\
&\quad + \sum_{l=-\frac{D}{2}+1}^{\infty} a_l (-)^{-m+l+\frac{D}{2}} B\Big(n - \frac{d-1}{2}, \frac{d}{2} - n\Big)\Bigg\}
\end{aligned}
$$

where $m(d), m'(d), b_l(d)$ and $b_l'(d)$ are all evaluated at $d = D$, the d-dependency being suppressed for the sake of clarity. Although in evaluating diagrams, d-dependencies elsewhere will contribute still more finite terms, the above expression already makes abundantly clear that for any nontrivial $f^{(k)}(\lambda)$, evaluation of the finite part using this method will be extremely difficult if not impossible.

Although the result above is the primary ingredient in their evaluation, the expressions relevant to the main text are still one step removed from it and will be discussed in the next section.

B.3 Anti-de Sitter integrals

The diverging integrals encountered in the main body of this thesis are of just a few different types and can be classified by their tensor character and order of divergence. No attempt will be made to evaluate their finite parts.

First, a simple logarithmic divergence for $d \to 4$ can be computed directly from (B.2):

$$\begin{aligned} &\operatorname*{Pole}_{d\to 4} \int_{\mathcal{N}} \mathrm{d}^d x (-\lambda_F(y;x))^{2-d} f(x) \\ = \quad & f(y) \operatorname*{Pole}_{d\to 4} \int_{\mathcal{N}} \mathrm{d}^d x (-\lambda_F(y;x))^{2-d} \\ = \quad & -iR^d \pi^{\frac{d}{2}} 2 f(y) \frac{1}{4-d} \end{aligned} \tag{B.4}$$

where $D = 4$ has been replaced by d again to keep some sense of the d dependence of the result.

For a quadratically diverging integral

$$\begin{aligned} &\operatorname*{Pole}_{d\to 4} \int_{\mathcal{N}} \mathrm{d}^d x (-\lambda_F(y;x))^{1-d} f(x) \\ = \quad & \operatorname*{Pole}_{d\to 4} \int_{\mathcal{N}} \mathrm{d}^d x (-\lambda_F(y;x))^{1-d} [f(y) + R\hat{\varepsilon}^M (\partial_M f)(y) \\ & \qquad + \frac{R^2}{2} \hat{\varepsilon}^M \hat{\varepsilon}^N (\partial_M \partial_N f)(y) + \ldots] \end{aligned}$$

the first and second order derivatives of $f(x)$ need to be taken into consideration as well. The first order derivative term may appear to be odd and therefore vanish but does, in truth, not[4]. Examining the case where $x(\vec{r}, t)$

[4]This is not in agreement with Filthaut [Fil89], Appendix A, and Filthaut and Dullemond [FD91], where the first order derivative term and the $M \neq N$ parts of the second order term are discarded as "odd". Because of this they wind up with no curvature terms and the operator $\Box = \eta^{MN} \partial_M \partial_N$ instead of $P^{MN} \partial_M \partial_N$ — a fact which, fortunately, does not invalidate their final results because only scalar fields are involved. Moreover in [Fil89] there is a sign error in the expression for the second order derivative which appears to be corrected in the article.

is an arbitrary point on the hypersurface $\mathcal{N}$ in $\mathcal{M}$ (2.25) and y is the origin $y(0) = (\vec{0}, 0, R)$ on that hypersurface (2.39),

$$\hat{\varepsilon} = \hat{x} - \hat{y}(0) = (\frac{\vec{r}}{R}, \sqrt{1 + \frac{r^2}{R^2}} \sin\frac{t}{R}, \sqrt{1 + \frac{r^2}{R^2}} \cos\frac{t}{R} - 1)$$

it is clear that all components transverse to $y(0)$ are odd in $x_{\mathcal{N}} = (\vec{r}, t)$ while the parallel one is even. By extension the same is true for arbitrary y:

$$\hat{\varepsilon}^M = \left(P^{MN}(y) + \hat{y}^M \hat{y}^N\right) \hat{\varepsilon}_N \text{ where } \begin{cases} P^{MN}(y)\hat{\varepsilon}_N \text{ is odd in } x_{\mathcal{N}} \\ \hat{y}^M \hat{y}^N \hat{\varepsilon}_N \text{ is even in } x_{\mathcal{N}} \end{cases} \tag{B.5}$$

so that for arbitrary $g(\lambda)$

$$\int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{\varepsilon}^M = \hat{y}^M \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{y} \cdot \hat{\varepsilon} = -\frac{1}{2}\hat{y}^M \int_{\mathcal{N}} \mathrm{d}^d x\, g(-\lambda)\hat{\varepsilon}^2. \tag{B.6}$$

The second order derivative part yields an integral of the form

$$\begin{aligned} &\int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{\varepsilon}^M \hat{\varepsilon}^N \\ &\quad = \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda) \left(P^M{}_P(y) + \hat{y}^M \hat{y}_P\right) \hat{\varepsilon}^P \left(P^N{}_Q(y) + \hat{y}^N \hat{y}_Q\right) \hat{\varepsilon}^Q \\ &\quad = P^M{}_P(y) P^N{}_Q(y) \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{\varepsilon}^P \hat{\varepsilon}^Q + \frac{1}{4}\hat{y}^M \hat{y}^N \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)(\hat{\varepsilon}^2)^2. \end{aligned}$$

The first term vanishes for $P \neq Q$ because $P^M{}_P(y)\hat{\varepsilon}^P$ is odd in $x_{\mathcal{N}}$; moreover, if the integral is evaluated for $y = y_0 = y(0)$ it also vanishes for $P = Q = d + 1$. The isotropy of $\mathcal{N}$ implies that the d other choices for P, Q yield identical contributions, which means

$$\begin{aligned} &P^M{}_P(y_0) P^N{}_Q(y_0) \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{\varepsilon}^P \hat{\varepsilon}^Q \\ &\quad = \sum_{P=1}^{d} P^M{}_P(y_0) P^N{}_P(y_0) \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{\varepsilon}^P \hat{\varepsilon}^P \\ &\quad = P^{MN}(y_0) \frac{P_{PQ}(y_0)}{d} \int_{\mathcal{N}} \mathrm{d}^d x\, g(\lambda)\hat{\varepsilon}^P \hat{\varepsilon}^Q. \end{aligned}$$

Manifest SO($d-1,2$) invariance ensures that this result, too, can be transported back to arbitrary y. Realizing $P_{PQ}\hat{\varepsilon}^P\hat{\varepsilon}^Q = \lambda$ (eqn. 2.36),

$$\int_{\mathcal{N}} \mathrm{d}^d x\ g(\lambda)\hat{\varepsilon}^M\hat{\varepsilon}^N = \frac{1}{d}P^{MN}(y)\int_{\mathcal{N}} \mathrm{d}^d x\ g(\lambda)\lambda + \frac{1}{4}\hat{y}^M\hat{y}^N\int_{\mathcal{N}} \mathrm{d}^d x\ g(\lambda)(\hat{\varepsilon}^2)^2. \tag{B.7}$$

Given that $\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda(y;x))^{1-d}(\hat{\varepsilon}^2)^2$ is finite and that $\varepsilon^2 = \lambda + \mathcal{O}(\lambda^2)$ (eqn. 2.38), the quadratically diverging case evaluates to

$$\begin{aligned}
&\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{1-d}f(x)\\
&= f(y)\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{1-d}\\
&\quad -\frac{1}{2}R(\hat{y}\cdot\partial f)(y)\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{1-d}\hat{\varepsilon}^2\\
&\quad -\frac{1}{2d}R^2P^{MN}(y)(\partial_M\partial_N f)(y)\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{2-d}\\
&= -iR^d\pi^{\frac{d}{2}}\left[-(1-R\hat{y}\cdot\partial)-\frac{R^2}{d}P^{MN}(y)\partial_M\partial_N\right]f(y)\frac{1}{4-d}.
\end{aligned} \tag{B.8}$$

A logarithmically diverging tensor integral

$$\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{1-d}\hat{\varepsilon}^M\hat{\varepsilon}^N f(x)$$

is of the form (B.7) and evaluates to

$$\begin{aligned}
&= -\frac{1}{d}P^{MN}f(y)\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{2-d}\\
&= -iR^d\pi^{\frac{d}{2}}\left[-\frac{2}{d}P^{MN}(y)\right]f(y)\frac{1}{4-d}.
\end{aligned} \tag{B.9}$$

Last but certainly not least consider its quadratically diverging cousin, where the first three terms in the Taylor expansion of $f(x)$ need to be taken into account again:

$$\operatorname*{Pole}_{d\to4}\int_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{-d}\hat{\varepsilon}^M\hat{\varepsilon}^N f(x)$$

$$= \operatorname*{Pole}_{d\to 4} \int_{\mathcal{N}} \mathrm{d}^d x (-\lambda_F(y;x))^{-d} \hat{\varepsilon}^M \hat{\varepsilon}^N [f(y) + R\hat{\varepsilon}^P (\partial_P f)(y) + \frac{R^2}{2}\hat{\varepsilon}^P \hat{\varepsilon}^Q (\partial_P \partial_Q f)(y) + \ldots]$$

Along the lines of (B.6) and (B.7), integrals with 3 and 4 tensor indices can be identified and for every index split in odd and even parts (B.5):

$$\begin{aligned}
&\int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\hat{\varepsilon}^M \hat{\varepsilon}^N \hat{\varepsilon}^P \\
&= -\frac{1}{8}\hat{y}^M \hat{y}^N \hat{y}^P \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\hat{(}\varepsilon^2)^3 \\
&\quad -\frac{1}{2}\Big(P^M{}_K(y)P^N{}_L(y)\hat{y}^P + P^M{}_K(y)\hat{y}^N P^P{}_L(y) \\
&\qquad +\hat{y}^M P^N{}_K(y)P^P{}_L(y)\Big) \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\hat{\varepsilon}^2 \hat{\varepsilon}^K \hat{\varepsilon}^L \\
&= -\frac{1}{8}\hat{y}^M \hat{y}^N \hat{y}^P \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\hat{(}\varepsilon^2)^3 \\
&\quad -\frac{1}{2d}\Big(P^{MN}(y)\hat{y}^P + P^{MP}(y)\hat{y}^N + P^{NP}(y)\hat{y}^M\Big) \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\hat{\varepsilon}^2\lambda
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
&\int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\hat{\varepsilon}^M \hat{\varepsilon}^N \hat{\varepsilon}^P \hat{\varepsilon}^Q \\
&= \frac{1}{16}\hat{y}^M \hat{y}^N \hat{y}^P \hat{y}^Q \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)(\hat{\varepsilon}^2)^4 \\
&\quad +\frac{1}{4d}\Big(P^{MN}(y)\hat{y}^P \hat{y}^Q \\
&\qquad +11 \text{ other permutations of } MNPQ\Big) \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)(\hat{\varepsilon}^2)^2\lambda \\
&\quad +\frac{1}{d(d+2)}\Big(P^{MN}(y)P^{PQ}(y) + P^{MP}(y)P^{NQ}(y) \\
&\qquad +P^{MQ}(y)P^{NP}(y)\Big) \int_{\mathcal{N}} \mathrm{d}^d x\; g(\lambda)\lambda^2.
\end{aligned} \tag{B.11}$$

Only the $g(\lambda)\lambda^2$ terms of above expressions survive in the pole part of the integration at hand, so together with (B.7)

$$\begin{aligned}
&\operatorname*{Pole}_{d\to4}\int\limits_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{-d}\hat{\varepsilon}^M\hat{\varepsilon}^N f(x)\\
&= -\frac{P^{MN}(y)}{d} f(y)\operatorname*{Pole}_{d\to4}\int\limits_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{1-d}\\
&\quad+\left[\frac{1}{4}\hat{y}^M\hat{y}^N-\frac{R}{2d}\left(P^{MN}(y)\hat{y}\cdot\partial+\hat{y}^{\{M}\mathcal{D}^{N\}}\right)\right.\\
&\quad\left.+\frac{R^2}{2d(d+2)}\left(P^{MN}(y)P^{KL}(y)+2P^{MK}(y)P^{NL}(y)\right)\partial_K\partial_L\right]f(y)\\
&\qquad\times\operatorname*{Pole}_{d\to4}\int\limits_{\mathcal{N}} \mathrm{d}^d x(-\lambda_F(y;x))^{2-d}\\
&= -iR^d\pi^{\frac{d}{2}}\left[\frac{1}{d}P^{MN}(y)(1-R\hat{y}\cdot\partial)+\frac{1}{2}\hat{y}^M\hat{y}^N-\frac{R}{d}\hat{y}^{\{M}\mathcal{D}^{N\}}\right.\\
&\quad\left.+\frac{R^2}{d(d+2)}\left(P^{MN}(y)P^{KL}(y)+2P^{MK}(y)P^{NL}(y)\right)\partial_K\partial_L\right]f(y)\\
&\qquad\times\frac{1}{4-d}
\end{aligned}\tag{B.12}$$

where $\mathcal{D}_N \stackrel{\text{def}}{=} P^M{}_N\partial_M$ (eqn. 2.31) and $\hat{y}^{\{M}\mathcal{D}^{N\}} \stackrel{\text{def}}{=} \hat{y}^M\mathcal{D}^N+\hat{y}^N\mathcal{D}^M$.

This completes the list of integral expressions encountered in this manuscript. It is evident that higher order divergences will involve more and more terms from the expansion of $f(y)$ and complexity will be increasing rapidly. Computing the finite part of these integrals appears both highly interesting and highly intractable.

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samenvatting

De theoretische hoge-energie fysica, die zich bezighoudt met het gedrag van de meest elementaire deeltjes, heeft al zo'n drie decennia last van zijn eigen succes. Het *Standaardmodel*, een quantumveldentheorie die indertijd geformuleerd is om alle bekende deeltjes en hun wisselwerkingen (sterk, zwak en electromagnetisch) te beschrijven, klopt namelijk onwaarschijnlijk goed met experimentele meetgegevens.

Onwaarschijnlijk, omdat de vierde en laatste bekende wisselwerking, de zwaartekracht, niet goed in het model past. Naïeve pogingen om de gangbare zwaartekrachtstheorie – Einstein's Algemene Relativiteitstheorie – te combineren met het Standaardmodel eindigen steevast in een niet werkende theorie. De oneindigheden, die erin voorkomen, zijn niet te interpreteren (de theorie is niet *renormaliseerbaar*). Soortgelijke oneindigheden komen ook in het Standaardmodel voor, maar daar kunnen ze met *regularisatie* en renormalisatie op een zinnige manier worden geïnterpreteerd. In de combinatie van de twee theorieën is dat niet langer mogelijk. Blijkbaar is er een onderliggende theorie die méér is dan zomaar de optelsom van het Standaardmodel en de Algemene Relativiteitstheorie. Maar zolang de voorspellingen van het Standaardmodel perfect kloppen met de waarnemingen, is er geen enkele experimentele onderbouwing voor speculaties over deze overkoepelende theorie (Grand Unified Theory oftewel *GUT*). Gelukkig begint hier recentelijk verandering in te komen met waarnemingen van neutrino-massa en zelfs mogelijk van het Higgs-deeltje, waarvan het bestaan door het Standaardmodel wordt voorspeld maar waarvan de massa een stuk lager zou kunnen uitpakken dan op grond van het model verwacht kan worden.

Het gebrek aan meetgegevens heeft het enthousiasme waarmee theoretisch fysici zich op mogelijke nieuwe theorieën hebben gestort nauwelijks aangetast. IJkgravitatie, supersymmetrische uitbreidingen op het Standaardmodel, superstrings, elke denkbare mogelijkheid is verkend. De stringtheorieën lijken daarvan de meest veelbelovende kandidaten op te leveren.

Desalniettemin blijven zulke theorieën speculatief en complex. Het kan daarom zinnig zijn om niet direct een nieuwe theorie te formuleren, maar te beschouwen wat er gebeurt als de zwaartekracht voorzichtig in het Standaardmodel wordt geïntroduceerd. Niet als een dynamisch veld, dat volledig meedoet aan alle interacties – dit leidt direct tot de al genoemde oneindigheden – maar als een achtergrond waartegen het spel van elementaire deeltjes zich afspeelt. Dit kan een idee geven van de eerste zwaartekrachtgerelateerde correcties op bekende fysische grootheden zoals het anomaal magnetisch moment van het electron. Dat is het uiteindelijke doel van het onderzoeksprogramma waarvan dit proefschrift een klein onderdeel is.

Einstein leert dat de zwaartekracht geen traditionele kracht is, maar het gevolg van een intrinsieke kromming in de ruimte en tijd waarin we leven. Massa kromt de ruimte om zich heen en beïnvloedt op die manier zijn omgeving. Fysica in een zwaartekrachtveld is dus fysica in een gekromde ruimte. Hoe een quantumveldentheorie in een gekromde ruimte moet worden geformuleerd is al tientallen jaren bekend; het Standaardmodel in zo'n ruimte kan zo worden opgeschreven. Hier iets praktisch mee uitrekenen is een heel ander verhaal. Een willekeurige kromming maakt het probleem veel te complex. Er moet vereenvoudigd worden.

De meest eenvoudige gekromde ruimte is een *maximaal symmetrische ruimte* die er op elk tijdstip, vanuit elk punt, hetzelfde uitziet. Er zijn maar drie verschillende typen van: *de Sitter* ruimte, "normale" platte *Minkowski* ruimte, en *anti-de Sitter* ruimte. Vanuit natuurkundig oogpunt is de laatste de interessantste; de platte ruimte is bekend gebied, en de de Sitter ruimte heeft geen ondergrens aan de energie, waardoor een deeltje naar steeds lagere energiewaarden kan vervallen alsof het in een bodemloze put valt. In de anti-de Sitter ruimte zijn veel zaken weliswaar een stuk ingewikkelder, maar nog steeds door te rekenen en consistent.

In het kader van een promotie-onderzoek is het ook noodzakelijk de breedte af te bakenen. De quantumelectrodynamica of *QED* uit de titel van het proefschrift maakt onderdeel uit van het Standaardmodel. Het beschrijft de wisselwerking tussen het electromagnetisch veld, dat de drager is van onder meer radiogolven en licht (maar ook van b.v. moleculaire bindingen), en geladen deeltjes als het electron.

Het proefschrift valt uiteen in drie delen. Een aantal meer en minder bekende eigenschappen van de anti-de Sitter ruimte en de gebruikte technieken zijn verzameld in hoofdstuk 2. Vervolgens worden in hoofdstuk 3 de meest relevante *propagatoren* uitgewerkt. Deze propagatoren beschrijven de ontwikkeling van een (quantum)veld in de tijd; de spinorpropagator beschrijft o.m. het electron, de vectorpropagator het electromagnetisch veld. De eerste propagator, de scalarpropagator, wordt niet direct gebruikt maar vormt het startpunt voor de spinorpropagator. De uitwerking is gebaseerd op die van Dullemond en van Beveren [DvB85] en Filthaut [Fil89]. De spinorpropagator is een generalisatie naar $d \neq 4$ dimensies van het werk van Janssen en Dullemond [JD86], de vectorpropagator een generalisatie van het werk van dezelfde auteurs en een verdere uitwerking van de afstudeerscriptie van de schrijver [dH92]. De uitbreiding naar d dimensies helpt bij de regularisatie van oneindigheden in de berekeningen van het volgende hoofdstuk.

Hoofdstuk 4 is de kern van het proefschrift. Hier worden de drie elementaire QED processen doorgerekend tegen een anti-de Sitter achtergrond;

deels vormt dit een veldtest van de technieken die voor de ϕ^4 "toy theory" zijn gebruikt door Filthaut en Dullemond [FD91]. Deze processen zijn de vacuumpolarisatie, die beschrijft hoe een foton van het electromagnetisch veld omgeven is door een wolk van (virtuele) electronen; de zelf-energie van een electron omgeven door een wolk van fotonen; en het vertexdiagram dat een eerste correctie vormt op de interactie tussen het electron en het electromagnetisch veld. Een belangrijke beperking van de gebruikte technieken is dat alleen het *oneindige* deel van de processen kan worden uitgerekend. Er kunnen daarmee uitspraken worden gedaan over de renormaliseerbaarheid en daarmee de fysische levensvatbaarheid van QED in de anti-de Sitter ruimte, maar meetbare grootheden als het genoemde anomaal magnetisch moment kunnen (nog) niet worden berekend. In de conclusies worden mogelijkheden verkend om hier verandering in te brengen.

De belangrijkste conclusie is dat de genoemde drie processen zich in grote lijnen net zo gedragen in de anti-de Sitter ruimte als in de platte ruimte, en de renormalisatie van de theorie geen fundamentele problemen met zich meebrengt. De berekeningen zijn het startpunt voor een verdere uitwerking van aan de ene kant hogere-orde correcties, waarvan verwacht kan worden dat de kromming van de ruimte hieraan nieuwe bijdragen levert, en aan de andere kant de eindige delen, waarmee de invloed van deze kromming op de fysica kan worden bepaald. Vervolgens zou hetzelfde spel kunnen worden gespeeld voor de rest van het Standaardmodel. Als sommige van de voorspellingen experimenteel verifieerbaar zouden blijken, levert dat waardevolle informatie op over de manier waarop de werkelijkheid in elkaar zit als de voorspelling klopt — en nog veel meer als deze niet klopt. Kortom, er is nog genoeg te doen.

curriculum vitae

De schrijver is geboren op 21 september 1967 in Voorburg. In 1985 behaalde hij zijn VWO-diploma aan het Nederrijn College te Arnhem en ging natuurkunde studeren aan de Katholieke Universiteit Nijmegen. In 1986 behaalde hij hier zijn propedeuse. Na een adempauze (of, wat meer trendy, “gap year”) van ongeveer anderhalf jaar heeft hij onder begeleiding van prof. dr. C. Dullemond gewerkt aan anti-de Sitter vectorpropagatoren in d dimensies, resulterend in het doctoraaldiploma in 1992. Het jaar daarop werd dit voortgezet in een aanstelling als AIO onder dezelfde hoogleraar.

Na afloop van het AIO-contract eind 1996 verhuisden het laatste stuk onderzoek en het schrijven van het proefschrift naar de avonduren, omdat de dagen gevuld werden met software-ontwikkeling: eerst Unify Vision bij Computer- en CommunicatieZaken op de universiteit, vanaf 1999 Java en Vision bij Objectivity Ltd. in Groot-Brittannië. Het heeft daarmee even geduurd, maar het resultaat van het onderzoek ligt nu voor u.